

# EXISTENCE OF NEUMANN AND SINGULAR SOLUTIONS OF THE FAST DIFFUSION EQUATION

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**ABSTRACT.** Let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ ,  $0 < m \leq \frac{n-2}{n}$ ,  $a_1, a_2, \dots, a_{i_0} \in \Omega$ ,  $\delta_0 = \min_{1 \leq j \leq i_0} \text{dist}(a_j, \partial\Omega)$  and let  $\Omega_\delta = \Omega \setminus \bigcup_{i=1}^{i_0} B_\delta(a_i)$  and  $\hat{\Omega} = \Omega \setminus \{a_1, \dots, a_{i_0}\}$ . For any  $0 < \delta < \delta_0$  we will prove the existence and uniqueness of positive solution of the Neumann problem for the equation  $u_t = \Delta u^m$  in  $\Omega_\delta \times (0, T)$  for some  $T > 0$ . We will prove the existence of singular solutions of this equation in  $\hat{\Omega} \times (0, T)$  for some  $T > 0$  that blow-up at the points  $a_1, \dots, a_{i_0}$ .

## 1. INTRODUCTION

Recently there is a lot of research on the equation

$$u_t = \Delta u^m \quad (1.1)$$

by M. Bonforte, E. Chasseigne, M. Fila, G. Grillo, J.L. Vazquez, M. Winkler, E. Yanagida [BGV1], [BGV2], [BV1], [BV2], [BV3], [CV], [FVWY], P. Daskalopoulos, M. Del Pino and N. Sesum [DPS], [DS1], [DS2], S.Y. Hsu [Hs2-3], K.M. Hui [Hu2-3], M. Del Pino and M. Sáez [PS], L.A. Peletier and H. Zhang [PZ], etc. This equation arises in many physical models. When  $m > 1$ , it is called the porous medium which models the diffusion of gases through porous media [A]. When  $m = 1$ , (1.1) is the heat equation. When  $0 < m < 1$ , it is usually called the fast diffusion equation. When  $m = \frac{n-2}{n+2}$ , (1.1) appears in the study of Yamabe flow on  $\mathbb{R}^n$ . In fact the metric  $g_{ij} = u^{\frac{4}{n+2}} dx^2$ ,  $u > 0$ , is a solution of the Yamabe flow [DS2], [PS],

$$\frac{\partial g_{ij}}{\partial t} = -R g_{ij}$$

in  $\mathbb{R}^n$ ,  $n \geq 3$ , if and only if  $u$  is a solution of

$$u_t = \frac{n-1}{m} \Delta u^m$$

with  $m = \frac{n-2}{n+2}$  where  $R$  is the scalar curvature of  $g_{ij}$ . We refer the readers to the book [V3] by J.L. Vazquez for the basics of (1.1) and the books [DK], [V2], by P. Daskalopoulos, C.E. Kenig and J.L. Vazquez for the most recent results of (1.1). We also refer to the paper [BV3], by M. Bonforte and J. L. Vazquez for the non local version of (1.1).

As observed by L. Peletier [P] and J.L Vazquez [V1] there is a big difference on the behaviour of solutions of (1.1) for  $(n-2)/n < m < 1$ ,  $n \geq 3$ , and for  $0 < m \leq (n-2)/n$ ,  $n \geq 3$ . For example there is a  $L^1 - L^\infty$  regularizing effect for the solutions of

$$\begin{cases} u_t = \Delta u^m, u \geq 0, & \text{in } \mathbb{R}^n \times (0, T) \\ u(x, 0) = u_0 & \text{in } \mathbb{R}^n \end{cases} \quad (1.2)$$

with  $0 \leq u_0 \in L^1_{loc}(\mathbb{R}^n)$  for any  $(n-2)/n < m < 1$  [HP], [DaK]. However there is no such  $L^1 - L^\infty$  regularizing effect for solutions of (1.2) when  $0 < m \leq (n-2)/n$ ,  $n \geq 3$ , [V2]. When  $\frac{n-2}{n} < m < 1$ ,

existence and uniqueness of global weak solution of (1.2) for any  $0 \leq u_0 \in L^1_{loc}(\mathbb{R}^n)$  has been proved by M.A. Herrero and M. Pierre in [HP]. When  $0 < m \leq (n-2)/n$  and  $n \geq 3$ , existence of positive smooth solutions of (1.2) for any  $0 \leq u_0 \in L^p_{loc}(\mathbb{R}^n)$ ,  $p > \max(1, (1-m)n/2)$ , satisfying the condition,

$$\liminf_{R \rightarrow \infty} \frac{1}{R^{n-\frac{2}{1-m}}} \int_{|x| \leq R} u_0 dx \geq C_1 T^{\frac{1}{1-m}} \quad (1.3)$$

for some constant  $C_1 > 0$  is proved by S.Y. Hsu in [Hs3].

In this paper we will study the existence of singular solutions of (1.1). Study of singular solutions of nonlinear elliptic equations were also obtained by H. Brezis and L. Veron [BrV], B. Gidas and J. Spruck [GS], etc. In order to study the singular solutions of (1.1) we will first prove the existence of positive smooth solution of the Neumann problem for (1.1) in smooth bounded domains with a finite numbers of holes when  $0 < m \leq (n-2)/n$ ,  $n \geq 3$ . When  $n \geq 3$  and  $m = (n-2)/n$ , we will prove the existence of singular solutions of (1.1) in a smooth bounded domain that blow-up at a finite number of points in the domain. More precisely let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^n$ ,  $n \geq 1$ ,  $a_1, a_2, \dots, a_{i_0} \in \Omega$ ,  $\delta_0 = \min_{1 \leq i, j \leq i_0} (\text{dist}(a_i, \partial\Omega), |a_i - a_j|)/2$ ,  $\Omega_\delta = \Omega \setminus \bigcup_{i=1}^{i_0} B_\delta(a_i)$ ,  $\hat{\Omega} = \Omega \setminus \{a_1, \dots, a_{i_0}\}$  and  $\mathbb{R} = \mathbb{R} \setminus \{a_1, \dots, a_{i_0}\}$ . We will prove the following three main theorems.

**Theorem 1.1.** *Let  $n \geq 3$ ,  $0 < m \leq \frac{n-2}{n}$ ,  $0 < \delta < \delta_0$ ,  $0 \leq u_0 \in L^p(\Omega_\delta)$  for some constant  $p > \frac{n(1-m)}{2}$ ,  $0 \leq f \in L^\infty_{loc}(\partial\Omega \times [0, \infty))$  and  $0 \leq g_i \in L^\infty_{loc}(\partial B_\delta(a_i) \times [0, \infty))$  for all  $i = 1, \dots, i_0$ . Suppose either  $u_0 \not\equiv 0$  on  $\Omega_\delta$  or*

$$\int_0^t \int_{\partial\Omega} f d\sigma ds + \sum_{i=1}^{i_0} \int_0^t \int_{\partial B_\delta(a_i)} g_i d\sigma ds > 0 \quad \forall t > 0.$$

*Then there exists a unique solution  $u$  for the equation*

$$\begin{cases} u_t = \Delta u^m & \text{in } \Omega_\delta \times (0, \infty) \\ \frac{\partial u^m}{\partial \nu} = f & \text{on } \partial\Omega \times (0, \infty) \\ \frac{\partial u^m}{\partial \nu} = g_i & \text{on } (\bigcup_{i=1}^{i_0} \partial B_\delta(a_i)) \times (0, \infty) \quad \forall i = 1, \dots, i_0 \\ u(x, 0) = u_0(x) & \text{in } \Omega_\delta \end{cases} \quad (1.4)$$

*that satisfies*

$$\int_{\Omega_\delta} u(x, t) dx = \int_{\Omega_\delta} u_0 dx + \int_0^t \int_{\partial\Omega} f d\sigma ds + \sum_{i=1}^{i_0} \int_0^t \int_{\partial B_\delta(a_i)} g_i d\sigma ds \quad \forall t > 0 \quad (1.5)$$

*where  $\frac{\partial}{\partial \nu}$  is the derivative on  $\partial\Omega_\delta$  with respect to the unit outward normal of the domain  $\Omega_\delta$ . Moreover if  $f \equiv 0$  on  $\partial\Omega \times (0, \infty)$  and  $g_i$ ,  $i = 1, \dots, i_0$ , are nonnegative monotone decreasing functions of  $t > 0$ , then*

$$u_t \leq \frac{u}{(1-m)t} \quad (1.6)$$

*in  $\Omega_\delta \times (0, \infty)$ .*

**Theorem 1.2.** *Let  $n \geq 3$ ,  $0 < m \leq \frac{n-2}{n}$ ,  $p > \frac{n(1-m)}{2}$ ,  $0 \leq f \in L^\infty_{loc}(\partial\Omega \times [0, \infty))$ . Let  $0 \leq u_0 \in L^p_{loc}(\hat{\Omega})$  be such that*

$$\frac{C_1}{|x - a_i|^q e^{\frac{1}{\delta_1^2 - |x - a_i|^2}}} \leq u_0(x) \leq \frac{C_2}{|x - a_i|^q} \quad \forall 0 < |x - a_i| \leq \delta_1, i = 1, \dots, i_0 \quad (1.7)$$

for some constants  $C_1 > 0$ ,  $C_2 > 0$ ,  $q \geq \max\left(\frac{n}{2m}, \frac{n-2}{m}\right)$  and  $0 < \delta_1 < \min\left(\frac{(1-m)q}{4+(1-m)q}, \delta_0\right)$ . Then there exists a solution  $u$  of

$$\begin{cases} u_t = \Delta u^m & \text{in } \hat{\Omega} \times (0, \infty) \\ \frac{\partial u^m}{\partial \nu} = f & \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \hat{\Omega} \end{cases} \quad (1.8)$$

such that

$$u(x, t) \geq \frac{C_1}{|x - a_i|^q e^{\frac{1}{\delta_1^2 - |x - a_i|^2}}} \quad \forall 0 < |x - a_i| < \delta_1, t > 0, i = 1, \dots, i_0 \quad (1.9)$$

and

$$u(x, t) \leq \frac{C_T}{|x|^q} \quad \forall 0 < |x - a_i| \leq \frac{\delta_1}{2}, 0 < t \leq T, i = 1, \dots, i_0 \quad (1.10)$$

hold for some constant  $C_T > 0$  where  $\partial/\partial\nu$  is the derivative with respect to the unit outward normal on  $\partial\Omega$ .

**Theorem 1.3.** Let  $n \geq 3$ ,  $0 < m \leq \frac{n-2}{n}$ ,  $p > \frac{n(1-m)}{2}$ . Let  $0 \leq u_0 \in L_{loc}^p(\hat{\mathbb{R}})$  be such that (1.7) holds for some constants  $C_1 > 0$ ,  $C_2 > 0$ ,  $q \geq \max\left(\frac{n}{2m}, \frac{n-2}{m}\right)$  and

$$0 < \delta_1 < \min\left(\frac{(1-m)q}{4+(1-m)q}, \frac{1}{2} \min_{1 \leq i, j \leq i_0} |a_i - a_j|\right).$$

Then there exists a solution  $u$  of

$$\begin{cases} u_t = \Delta u^m & \text{in } \hat{\mathbb{R}} \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \hat{\mathbb{R}} \end{cases} \quad (1.11)$$

such that (1.9) and (1.10) hold for some constant  $C_T > 0$ .

The plan of the paper is as follows. In section two we will prove some a priori estimates for  $C^{2,1}$  solution of (1.4). In section three we will prove Theorem 1.1. In section four we will prove Theorem 1.2 and Theorem 1.3.

We start with some notations and definitions that will be used in this paper. Let  $\Omega_1 \subset \mathbb{R}^n$  be a smooth bounded domain and let  $\Sigma_1, \Sigma_2$  be relatively open subsets of  $\partial\Omega_1$  such that  $\partial\Omega_1 = \Sigma_1 \cup \Sigma_2$  and if  $n \geq 2$ , then  $\bar{\Sigma}_1 \cap \bar{\Sigma}_2$  is a  $C^2$  manifold of dimension  $n - 2$ . For any  $0 < m < 1$ ,  $0 \leq u_0 \in L^1(\Omega_1)$ ,  $f \in L_{loc}^1(\Sigma_1 \times [0, \infty))$  and  $g \in L_{loc}^1(\Sigma_2 \times [0, \infty))$ , we say that  $u$  is a very weak solution (subsolution, supersolution respectively) of

$$\begin{cases} u_t = \Delta u^m & \text{in } \Omega_1 \times (0, T) \\ \frac{\partial u^m}{\partial \nu} = f & \text{on } \Sigma_1 \times (0, T) \\ u = g & \text{on } \Sigma_2 \times (0, T) \\ u(x, 0) = u_0(x) & \text{in } \Omega_1 \end{cases} \quad (1.12)$$

if  $0 \leq u \in C([0, T]; L^1(\Omega_1))$  satisfies ( $\geq, \leq$  respectively)

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\Omega_1} (u \eta_t + u^m \Delta \eta) dx dt + \int_{t_1}^{t_2} \int_{\Sigma_1} f \eta d\sigma dt \\ & = \int_{t_1}^{t_2} \int_{\Sigma_2} g^m \frac{\partial \eta}{\partial \nu} d\sigma dt + \int_{\Omega_1} u(x, t_2) \eta(x, t_2) dx - \int_{\Omega_1} u(x, t_1) \eta(x, t_1) dx \end{aligned} \quad (1.13)$$

for any  $0 < t_1 < t_2 < T$ , and  $\eta \in C^2(\overline{\Omega}_1 \times (0, T))$  satisfying  $\eta = 0$  on  $\Sigma_2 \times (0, T)$ , and  $\partial\eta/\partial\nu = 0$  on  $\Sigma_1 \times (0, T)$  and  $u$  has initial value  $u_0$ . We say that  $u$  is a solution (subsolution, supersolution respectively) of (1.12) if  $u \in L_{loc}^\infty(\overline{\Omega}_1 \times (0, T))$  is positive in  $\Omega_1 \times (0, T)$  and satisfies (1.1) in  $\Omega_1 \times (0, T)$  ( $\leq, \geq$  respectively) in the classical sense with

$$\|u(\cdot, t) - u_0\|_{L^1(\Omega_\delta)} \rightarrow 0 \quad \text{as } t \rightarrow 0 \quad (1.14)$$

and also satisfies (1.13) ( $\geq, \leq$  respectively) for any  $0 < t_1 < t_2 < T$ , and  $\eta \in C^2(\overline{\Omega}_1 \times (0, T))$  satisfying  $\eta = 0$  on  $\Sigma_2 \times (0, T)$ , and  $\partial\eta/\partial\nu = 0$  on  $\Sigma_1 \times (0, T)$ .

We say that  $u$  is a solution (subsolution, supersolution respectively) of (1.8) if  $u \in L_{loc}^\infty((\overline{\Omega} \setminus \{a_1, \dots, a_{i_0}\}) \times (0, T)) \cap C^{2,1}((\overline{\Omega} \setminus \{a_1, \dots, a_{i_0}\}) \times (0, T))$  is positive in  $\hat{\Omega} \times (0, T)$  and satisfies (1.1) in  $\hat{\Omega} \times (0, T)$  ( $\leq, \geq$  respectively) in the classical sense with

$$\lim_{t \rightarrow 0} \int_{\hat{\Omega}} u(x, t) \eta(x) dx = \int_{\hat{\Omega}} u_0 \eta dx \quad \forall \eta \in C_0^\infty(\hat{\Omega}) \quad (1.15)$$

and also satisfies

$$\int_{t_1}^{t_2} \int_{\hat{\Omega}} (u \eta_t + u^m \Delta \eta) dx dt + \int_{t_1}^{t_2} \int_{\partial \Omega} f \eta d\sigma dt = \int_{\Omega_1} u(x, t_2) \eta(x, t_2) dx - \int_{\Omega_1} u(x, t_1) \eta(x, t_1) dx \quad (1.16)$$

( $\geq, \leq$  respectively) for any  $0 < t_1 < t_2 < T$ , and  $\eta \in C_0^2((\overline{\Omega} \setminus \{a_1, \dots, a_{i_0}\}) \times (0, T))$  satisfying  $\partial\eta/\partial\nu = 0$  on  $\partial\Omega \times (0, T)$ .

We say that  $u$  is a solution (subsolution, supersolution respectively) of (1.11) if  $u \in L_{loc}^\infty(\mathbb{R} \times (0, T)) \cap C^{2,1}(\mathbb{R} \times (0, T))$  is positive in  $\mathbb{R} \times (0, T)$  and satisfies (1.1) in  $\mathbb{R} \times (0, T)$  ( $\leq, \geq$  respectively) in the classical sense with

$$\lim_{t \rightarrow 0} \int_{\hat{\Omega}} u(x, t) \eta(x) dx = \int_{\hat{\Omega}} u_0 \eta dx \quad \forall \eta \in C_0^\infty(\mathbb{R}). \quad (1.17)$$

For any  $x_0 \in \mathbb{R}^n$ ,  $x'_0 \in \mathbb{R}^{n-1}$ ,  $R > 0$ , we let  $B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$ ,  $B'_R(x'_0) = \{x' \in \mathbb{R}^{n-1} : |x - x'_0| < R\}$ ,  $\hat{B}_R(x_0) = B_R(x_0) \setminus \{x_0\}$ ,  $B_R = B_R(0)$ ,  $\hat{B}_R = \hat{B}_R(0)$  and  $B'_R = B'_R(0)$ . For any  $a \in \mathbb{R}$ , we let  $a_+ = \max(a, 0)$  and  $a_- = \max(-a, 0)$ . For any set  $A \in \mathbb{R}^n$ , we let  $\chi_A$  be the characteristic function of the set  $A$ .

## 2. A PRIORI ESTIMATES

In this section we will prove some a priori estimates for the solutions of (1.4). We will also prove a  $L^p - L^\infty$  estimates for the solutions of (1.4) for some constant  $p > 1$ . These  $L^p - L^\infty$  estimates will be used in section three to give uniform upper bound for the approximating  $C^{2,1}$  solutions of (1.4) which appear in the construction of solution of (1.4). Note that similar  $L^\infty$  estimates are also obtained in [BV2], [CD], [CV], [D], [DK], [DGV1], [DGV2], [DK], [DKV] and [HP]. We first observe that by an argument similar to the proof of Lemma 2.3 of [DaK] we have the following result.

**Lemma 2.1** (cf. Lemma 1.1 of [Hs2]). *Let  $\Omega_1 \subset \mathbb{R}^n$  be a smooth bounded domain and let  $\Sigma_1, \Sigma_2$  be relatively open subsets of  $\partial\Omega_1$  such that  $\partial\Omega_1 = \Sigma_1 \cup \Sigma_2$  and if  $n \geq 2$  then  $\overline{\Sigma}_1 \cap \overline{\Sigma}_2$  is a  $C^2$  manifold of dimension  $n - 2$ . Let  $0 \leq u_{0,1}, u_{0,2} \in L^1(\Omega)$ ,  $f_1, f_2 \in L^1(\Sigma_1 \times (0, T))$  and  $g_1, g_2 \in C(\Sigma_2 \times (0, T))$  be such that  $0 \leq g_1 \leq g_2$  on  $\Sigma_2 \times (0, T)$ . Suppose  $u_1, u_2$  are subsolution and supersolution of (1.12) in  $\Omega_1 \times (0, T)$  with  $f = f_1, f_2, g = g_1, g_2$  and  $u_0 = u_{0,1}, u_{0,2}$ , respectively. Then*

$$\int_{\Omega_1} (u_1 - u_2)_+(x, t) dx \leq \int_{\Omega_1} (u_{0,1} - u_{0,2})_+(x, t) dx + \int_0^t \int_{\Sigma_1} (f_1 - f_2)_+ d\sigma ds \quad \forall 0 \leq t < T.$$

By the same argument as the proof of Lemma 3.4 of [Hu1] we have the following result.

**Lemma 2.2** (cf. Lemma 3.4 of [Hu1]). *Suppose  $\Omega \subset \mathbb{R}^n$  is a smooth bounded convex domain. For any  $x \in \partial\Omega$ ,  $x_0 \in \Omega$ , let  $n(x)$  be the unit outward normal vector at  $x$  with respect to  $\Omega$  and let  $n_1(x)$  be the unit vector along the line segment  $\overrightarrow{x_0x}$  from  $x_0$  to  $x$ . If  $\theta(x)$  is the angle between  $n(x)$  and  $n_1(x)$ , then there exists a constant  $0 < c_0 \leq 1$  such that*

$$0 < c_0 \leq \cos \theta(x) \leq 1 \quad \forall x \in \partial\Omega.$$

Now, we are going to prove some estimates for the solutions of (1.4).

**Lemma 2.3.** *Let  $n \geq 1$ ,  $0 < m < 1$ ,  $T > 0$ ,  $q \geq \frac{2}{1-m}$ ,  $0 < \delta_1 < \min\left(\frac{(1-m)q}{4+(1-m)q}, \delta_0\right)$ , and  $0 < \delta \leq \delta_2 < \left(\frac{(1-m)q}{4+(1-m)q}\right)\delta_1$ . Let  $0 \leq u_0 \in L^1(\overline{\Omega}_\delta)$  such that*

$$u_0(x) \leq \frac{C_2}{|x - a_i|^q} \quad \forall \delta \leq |x - a_i| < \delta_1, i = 1, \dots, i_0 \quad (2.1)$$

*holds for some constant  $C_2 > 0$ . Let  $f \in L^1(\partial\Omega \times (0, T))$  and  $g_i \in L^1(\partial B_\delta(a_i) \times (0, T))$ ,  $i = 1, 2, \dots, i_0$ , be such that  $\sup f < \infty$ ,  $\sup g_i < \infty$ , for all  $i = 1, 2, \dots, i_0$ . Let  $u$  be a solution of*

$$\begin{cases} u_t = \Delta u^m & \text{in } \Omega_\delta \times (0, T) \\ \frac{\partial u^m}{\partial \nu} = f & \text{on } \partial\Omega \times (0, T) \\ \frac{\partial u^m}{\partial \nu} = \frac{g_i}{\delta^{mq+1}} & \text{on } B_\delta(a_i) \times (0, T) \quad \forall i = 1, \dots, i_0 \\ u(x, 0) = u_0(x) & \text{in } \Omega_\delta. \end{cases} \quad (2.2)$$

*Then there exists a constant  $A_1 > 0$  such that*

$$u(x, t) \leq \phi_{A_1}(x - a_i, t) \quad \forall \delta \leq |x - a_i| < \delta_1, 0 \leq t < T, i = 1, \dots, i_0 \quad (2.3)$$

*holds for all  $0 < \delta \leq \delta_2$  where*

$$\phi_{A_1}(x, t) = \frac{A_1(1+t)^{\frac{1}{1-m}}}{|x|^q(\delta_1 - |x|)^{\frac{2}{1-m}}}. \quad (2.4)$$

*If  $\Omega$  is a smooth convex domain and*

$$\|u_0\|_{L^\infty(\overline{\Omega}_\delta)} \leq M_0 \quad (2.5)$$

*holds for some constant  $M_0 > 0$ , then there exists a constant  $M_1 > 0$  depending on  $M_0$  such that*

$$u(x, t) \leq M_1 \quad \forall (x, t) \in \overline{\Omega}_{\delta_2} \times [0, T) \quad (2.6)$$

*holds for any  $0 < \delta \leq \delta_2$ .*

*Proof.* We will use a modification of the proof of Lemma 1.3 of [Hs2] to prove the lemma. Without loss of generality it suffices to prove (2.3) for  $i = i_0 = 1$ . Let

$$A_1 = \max \left\{ C_2, \left( \frac{m(mq^2 + 2q + 2n + 4)}{1 - m} \right)^{\frac{1}{1-m}}, \left( \frac{2(\sup g_1)_+}{mq} \right)^{\frac{1}{m}} \right\} \quad (2.7)$$

and let  $\delta' \in (\delta_2, \delta_1)$ . For any  $0 < t_1 < t_2 < T$ , let

$$M = \max_{\substack{\delta' \leq |x - a_1| \leq \delta_1 \\ t_1 \leq t \leq t_2}} \frac{m|\nabla u|}{u^{1-m}}. \quad (2.8)$$

Since

$$\frac{\partial \phi_{A_1}^m}{\partial r}(x - a_1, t) = \frac{mA_1^m(1+t)^{\frac{m}{1-m}}}{r^{mq}(\delta_1 - r)^{\frac{2m}{1-m}}} \left( -\frac{q}{r} + \frac{2}{(1-m)(\delta_1 - r)} \right) \rightarrow \infty$$

uniformly on  $t \in [0, \infty]$  as  $r = |x - a_1| \rightarrow \delta_1^-$ , there exists a constant  $\delta'' \in (\delta', \delta_1)$  such that

$$\frac{\partial \phi_{A_1}^m}{\partial r}(x - a_1, t) > M \quad \forall \delta'' \leq |x - a_1| \leq \delta_1, t \geq 0. \quad (2.9)$$

Since  $q/\delta \geq 4/[(1-m)(\delta_1 - \delta)]$ , by direct computation,

$$\begin{aligned} \frac{\partial \phi_{A_1}^m}{\partial \nu}(x - a_1, t) &= \frac{mA_1^m(1+t)^{\frac{m}{1-m}}}{\delta^{mq}(\delta_1 - \delta)^{\frac{2m}{1-m}}} \left[ \frac{q}{\delta} - \frac{2}{(1-m)(\delta_1 - \delta)} \right] \\ &\geq \frac{mA_1^m q}{2\delta^{mq+1}(\delta_1 - \delta)^{\frac{2m}{1-m}}} \\ &\geq \frac{mA_1^m q}{2\delta^{mq+1}} \geq \frac{g_1}{\delta^{mq+1}} \quad \text{on } \partial B_\delta(a_1) \times (0, \infty). \end{aligned} \quad (2.10)$$

By (2.7),

$$\begin{aligned} \Delta \phi_{A_1}^m &= A_1^m(1+t)^{\frac{m}{1-m}} \left[ \frac{mq(mq - n + 2)}{|x|^{mq+2}(\delta_1 - |x|)^{\frac{2m}{1-m}}} - \frac{2m(2mq - n + 1)}{(1-m)|x|^{mq+1}(\delta_1 - |x|)^{\frac{1+m}{1-m}}} + \frac{2m(1+m)}{(1-m)^2|x|^{mq}(\delta_1 - |x|)^{\frac{2}{1-m}}} \right] \\ &\leq \frac{mA_1^m(mq^2 + 2q + 2n + 4)(1+t)^{\frac{m}{1-m}}}{(1-m)^2|x|^{mq+2}(\delta_1 - |x|)^{\frac{2}{1-m}}} \\ &\leq \phi_{A_1, t} \quad \forall 0 < |x| < \delta_1, t \geq 0. \end{aligned} \quad (2.11)$$

Hence by (2.9), (2.10) and (2.11) for any  $\delta'_1 \in (\delta'', \delta_1)$ ,  $\phi_{A_1}(x - a_1, t)$  is a supersolution of

$$\begin{cases} w_t = \Delta w^m & \text{in } (B_{\delta'_1}(a_1) \setminus B_\delta(a_1)) \times (0, T) \\ \frac{\partial w^m}{\partial \nu} = M & \text{on } \partial B_{\delta'_1}(a_1) \times (0, T) \\ \frac{\partial w^m}{\partial \nu} = \frac{g_1}{\delta^{mq+1}} & \text{on } \partial B_\delta(a_1) \times (0, T) \\ w(x, 0) = u_0(x) & \text{in } B_{\delta'_1}(a_1) \setminus B_\delta(a_1) \end{cases} \quad (2.12)$$

where  $\partial/\partial \nu$  is the derivative with respect to the unit outward normal at the boundary of the domain  $B_{\delta'_1}(a_1) \setminus B_\delta(a_1)$ . Since  $u$  is a subsolution of (2.12), by Lemma 2.1,  $\forall 0 < t_1 < t_2 < T$ ,  $\delta'_1 \in (\delta'', \delta_1)$ ,

$$\begin{aligned} &\int_{B_{\delta'_1}(a_1) \setminus B_\delta(a_1)} (u(x, t) - \phi_{A_1}(x - a_1, t))_+ dx \leq \int_{B_{\delta'_1}(a_1) \setminus B_\delta(a_1)} (u(x, t_1) - \phi_{A_1}(x - a_1, t_1))_+ dx \quad \forall t_1 \leq t \leq t_2 \\ \Rightarrow &\int_{B_{\delta_1}(a_1) \setminus B_\delta(a_1)} (u(x, t) - \phi_{A_1}(x - a_1, t))_+ dx \leq \int_{B_{\delta_1}(a_1) \setminus B_\delta(a_1)} (u(x, t_1) - \phi_{A_1}(x - a_1, t_1))_+ dx \quad \text{as } \delta'_1 \rightarrow \delta_1 \\ \Rightarrow &\int_{B_{\delta_1}(a_1) \setminus B_\delta(a_1)} (u(x, t) - \phi_{A_1}(x - a_1, t))_+ dx \leq 0 \quad \forall 0 < t < T \quad \text{as } t_1 \rightarrow 0, t_2 \rightarrow T \end{aligned}$$

and (2.3) follows.

Suppose now  $\Omega$  is a smooth convex domain and (2.5) holds for some constant  $M_0 > 0$ . Let  $n(x)$ ,

$n_1(x)$ ,  $\theta(x)$  and  $c_0$  be as in Lemma 2.2 with  $x_0 = a_1$ . By (2.3) there exists a constant  $M > 0$  such that

$$u(x, t) \leq M \quad \forall (x, t) \in \left( \bigcup_{i=1}^{i_0} \partial B_{\delta_2}(a_i) \right) \times (0, T)$$

holds for all  $0 < \delta \leq \delta_2$ . Let

$$w(x, t) = A_2(1+t)^{\frac{1}{1-m}} e^{\frac{|x-a_1|}{\delta_2}}$$

where

$$A_2 = \max \left\{ \left( \frac{m(1-m)(m+n-1)}{\delta_2^2} \right)^{\frac{1}{1-m}}, \left( \frac{\delta_2}{mc_0} (\sup f)_+ \right)^{\frac{1}{m}}, M, \|u_0\|_{L^\infty(\Omega_{\delta_2})} \right\}.$$

Then

$$w(x, 0) \geq u_0(x) \quad \text{in } \Omega_{\delta_2}$$

and

$$w(x, t) \geq u(x, t) \quad \text{on } \left( \bigcup_{i=1}^{i_0} \partial B_{\delta_2}(a_i) \right) \times (0, T).$$

By direct computation,

$$\begin{aligned} \Delta w^m &= \frac{mA_2^m}{\delta_2} \left[ \frac{m}{\delta_2} + \frac{n-1}{|x-a_1|} \right] (1+t)^{\frac{m}{1-m}} e^{\frac{m|x-a_1|}{\delta_2}} \leq \frac{mA_2^m(m+n-1)}{\delta_2^2} (1+t)^{\frac{m}{1-m}} e^{\frac{m|x-a_1|}{\delta_2}} \\ &\leq \frac{A_2}{1-m} (1+t)^{\frac{m}{1-m}} e^{\frac{|x-a_1|}{\delta_2}} = w_t, \quad \text{in } \Omega_{\delta_2} \times (0, T). \end{aligned}$$

Moreover by Lemma 2.2,

$$\frac{\partial w^m}{\partial \nu}(x, t) = \frac{\partial w^m}{\partial n_1}(x, t) \cdot \cos \theta(x) = \frac{mA_2^m}{\delta_2} \cos \theta(x) (1+t)^{\frac{m}{1-m}} e^{\frac{m|x-a_1|}{\delta_2}} \geq \frac{mA_2^m c_0}{\delta_2} \geq f(x, t)$$

for all  $x \in \partial\Omega$ ,  $0 < t < T$ . Hence by Lemma 2.1,

$$u(x, t) \leq w(x, t) \leq A_2(1+T)e^{M_2} \quad \text{in } \overline{\Omega}_{\delta_2} \times [0, T]$$

where  $M_2 = \max_{x \in \overline{\Omega}_{\delta_2}} \left( \frac{|x-a_1|}{\delta_2} \right)$  and (2.6) follows.  $\square$

**Lemma 2.4.** Let  $n \geq 1$ ,  $0 < m < 1$ ,  $0 < \delta < \delta_1 < \min(\delta_0, m/2)$ ,  $0 \leq f \in L_{loc}^\infty(\partial\Omega \times [0, \infty))$  and  $g_i \in L_{loc}^\infty([0, \infty))$ ,  $i = 1, \dots, i_0$ , be such that

$$\inf_{[0, \infty)} g_i(t) > m(q + 4\delta_1^{-2}) \quad \forall i = 1, \dots, i_0. \quad (2.13)$$

Let  $0 \leq u_0 \in L^1(\Omega_\delta)$  be such that

$$u_0(x) \geq \frac{C_1}{|x-a_i|^q e^{\frac{1}{\delta_1^2 - |x-a_i|^2}}} \quad \forall \delta \leq |x-a_i| < \delta_1, i = 1, \dots, i_0 \quad (2.14)$$

for some constants  $C_1 > 0$  and  $q \geq \max\left(\frac{n}{2m}, \frac{n-2}{m}\right)$  and let  $u$  be a solution of (2.2). Then

$$u(x, t) \geq \frac{C_1}{|x-a_i|^q e^{\frac{1}{\delta_1^2 - |x-a_i|^2}}} \quad \forall \delta \leq |x| < \delta_1, t > 0, i = 1, \dots, i_0 \quad (2.15)$$

holds for any  $0 < \delta \leq \delta_1/2$ .

*Proof.* Without loss of generality it suffices to prove the lemma when  $i_0 = 1$  and  $a_1 = 0$ . Let

$$\phi(x) = |x|^{-q}\psi(x), \quad \psi(x) = e^{-\frac{1}{\delta_1^2 - |x|^2}}.$$

By direct computation,

$$\begin{aligned} \Delta|x|^{-mq} &= mq(mq + 2 - n)|x|^{-mq-2} \geq 0, \\ \Delta\psi^m &= \frac{2m\psi^m}{(\delta_1^2 - |x|^2)^4} (2m|x|^2 - 4|x|^2(\delta_1^2 - |x|^2) - n(\delta_1^2 - |x|^2)^2) \end{aligned}$$

Hence

$$\begin{aligned} \Delta\phi^m &= \psi^m \Delta|x|^{-mq} + 2\nabla\psi^m \cdot \nabla|x|^{-mq} + |x|^{-mq} \Delta\psi^m \\ &\geq \frac{4m^2q|x|^{-mq}\psi^m}{(\delta_1^2 - |x|^2)^2} + |x|^{-mq} \Delta\psi^m \\ &\geq \frac{2m|x|^{-mq}\psi^m}{(\delta_1^2 - |x|^2)^4} ((2m - 4\delta_1^2)|x|^2 + (2mq - n)(\delta_1^2 - |x|^2)^2) \\ &\geq 0 \quad \text{in } B_{\delta_1}. \end{aligned} \tag{2.16}$$

By (2.13),

$$\left. \frac{\partial\phi^m}{\partial\nu} \right|_{|x|=\delta} = \frac{m\psi(\delta)^m}{\delta^{qm+1}} \left( q + \frac{2}{(\delta_1^2 - \delta^2)^2} \right) \leq \frac{g_i}{\delta^{qm+1}} \quad \forall 0 < \delta < \frac{\delta_1}{2}. \tag{2.17}$$

By (2.16) and (2.13) for any  $0 < \delta < \frac{\delta_1}{2}$ ,  $\phi$  is a subsolution of

$$\begin{cases} u_t = \Delta u^m & \text{in } B_{\delta_1} \times (0, \infty) \\ \frac{\partial u^m}{\partial\nu} = \frac{g_i}{\delta^{qm+1}} & \text{on } B_\delta \times (0, \infty) \\ u = 0 & \text{on } \partial B_{\delta_1} \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } B_{\delta_1} \setminus B_\delta. \end{cases} \tag{2.18}$$

Since  $u$  is a supersolution of (2.18), by Lemma 2.1 (2.15) follows.  $\square$

We will now prove a  $L^p - L^\infty$  estimates for the solution of (2.2). Since the proof is similar to the proof in section 1 of [Hu1], we will only sketch the argument here. For any smooth bounded domain  $\Omega \subset \mathbb{R}^n$ , let  $x_0 \in \partial\Omega$ . When  $n \geq 2$ , by rotation and translation of the coordinate axis we may assume that  $x_0$  is at the origin and the tangent plane to  $\partial\Omega$  at  $x_0$  is  $\mathbb{R}^{n-1} \times \{0\}$  and there exists a constant  $0 < R_0 \leq \delta_0/2$  and a smooth function

$$\phi_1 : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$$

with  $\phi_1 \in C_0^\infty(\mathbb{R}^{n-1})$ ,  $\phi_1(0) = 0$ ,  $\nabla\phi_1(0) = 0$ , and  $\sup_{x' \in \mathbb{R}^{n-1}} |\nabla\phi_1(x')| \leq \frac{1}{10}$  such that

$$\begin{cases} \Omega \cap B_{R_0} = \{(x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > \phi_1(x')\} \cap B_{R_0} \\ \partial\Omega \cap B_{R_0} = \{(x', x_n) : x' \in \mathbb{R}^{n-1}, x_n = \phi_1(x')\} \cap B_{R_0}. \end{cases}$$

Then  $(\cup_{i=1}^{i_0} B_{\delta_0/2}(a_i)) \cap B_{R_0}(x_0) = \phi$ . For any  $x = (x', x_n) \in \Omega \cap B_{R_0}$  with  $x' \in \mathbb{R}^{n-1}$ ,  $x_n \in \mathbb{R}$ , let

$$\psi_1(x) = (x', x_n - \phi_1(x')) \tag{2.19}$$

and  $\Omega'_1 = \psi_1(\Omega \cap B_{R_0})$ . Then  $\psi_1$  is a diffeomorphism between  $\Omega \cap B_{R_0}$  and  $\Omega'_1$ . Let

$$\tilde{u}(y, t) = u(y', y_n + \phi_1(y'), t) \quad \forall (y', y_n) \in \Omega'_1.$$



For  $0 < r < R_0/5$ , let

$$D'_r = \begin{cases} \{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |y'| < r, 0 < y_n < r\} & \text{if } n \geq 2 \\ \{y \in \mathbb{R} : |y - x_0| < r\} & \text{if } n = 1 \end{cases}$$

and  $D_r = \psi_1^{-1}(D'_r)$  where  $\psi_1$  is given by (2.19) if  $n \geq 2$  and  $\psi_1$  is the identity map if  $n = 1$ . For  $0 < r < \rho < R_0/5$  and  $0 < t_1 < t_2 < t_0 < T$ ,  $t_2 < t < T$ , we let  $S(t) = \psi_1^{-1}(D'_r) \times (t_2, t]$ ,  $R(t) = \psi_1^{-1}(D'_\rho) \times (t_1, t]$ ,  $S = S(t_0)$  and  $R = R(t_0)$ .

**Lemma 2.5.** *Let  $0 < \delta < \delta_0$ ,  $f \in L^1(\partial\Omega \times (0, T))$ , and  $g_i \in L^1(\partial B_\delta(a_i) \times (0, T))$ ,  $i = 1, 2, \dots, i_0$ . Suppose  $u \in C^{2,1}(\overline{\Omega}_\delta \times (0, T))$  is a solution of (1.4) in  $\Omega_\delta \times (0, T)$ . For any  $a > 0$ , let  $v = \max(u, a)$ . Then*

$$\begin{aligned} & \int_{\Omega_\delta} v(x, t_2) \phi(x, t_2) dx + \int_{t_1}^{t_2} \int_{\Omega} \nabla v^m \cdot \nabla \phi dx dt \\ & \leq \int_{\Omega_\delta} v(x, t_1) \phi(x, t_1) dx + \int_{t_1}^{t_2} \int_{\Omega} v \phi_t dx dt + \int_{t_1}^{t_2} \int_{\partial\Omega} f \phi d\sigma dt + \int_{t_1}^{t_2} \int_{\cup_{i=0}^{i_0} \partial B_\delta(a_i)} g_i \phi d\sigma dt \end{aligned} \quad (2.20)$$

holds for any  $0 < t_1 < t_2 < T$  and  $\phi \in C(\overline{\Omega}_\delta \times (0, T))$  such that the weak derivatives  $\phi_t, \nabla \phi$ , exist and belong to  $L^2(\Omega_\delta \times (0, T))$  and

$$\int_{\Omega_\delta} v_t \phi dx + \int_{\Omega} \nabla v^m \cdot \nabla \phi dx \leq \int_{\partial\Omega} f \phi d\sigma + \int_{\cup_{i=0}^{i_0} \partial B_\delta(a_i)} g_i \phi d\sigma \quad (2.21)$$

holds for any  $0 < t < T$  and  $\phi \in C(\overline{\Omega}_\delta \times (0, T))$  such that for any  $0 < t' < T$  the weak derivatives  $\nabla \phi(x, t')$  exists and belong to  $L^2(\Omega_\delta)$ .

*Proof.* By approximation it suffices to show that (2.20) holds for any  $\phi \in C^\infty(\overline{\Omega}_\delta \times (0, T))$ . Since the proof of the lemma is similar to the proof of Lemma 1.1 of [Hu1] we will only sketch the argument here. We choose a sequence of functions  $\{p_k\}_{k=1}^\infty \subset C^\infty(\mathbb{R})$ ,  $0 \leq p_k \leq 1$ ,  $p_k(s) = 0$  for all  $s \leq a + (2k)^{-1}$ ,  $p_k(s) = 1$  for all  $s \geq a + k^{-1}$ ,  $0 \leq p'_k(s) \leq Ck$  for some constant  $C > 0$ , and  $p_k(s) \rightarrow \chi_{(a, \infty)}(s)$  as  $k \rightarrow \infty$ . Multiplying (1.1) by  $p_k(u)\phi$  and integrating by parts,

$$\begin{aligned} & \int_{\Omega} q_k(u(x, t_2)) \phi(x, t_2) dx + \int_{t_1}^{t_2} \int_{\Omega} \nabla u^m \cdot \nabla \phi dx dt \\ & \leq \int_{\Omega} q_k(u(x, t_2)) \phi(x, t_2) dx + \int_{t_1}^{t_2} \int_{\Omega} \nabla u^m \cdot \nabla (p_k(u)\phi) dx dt \\ & \leq \int_{\Omega} q_k(u(x, t_1)) \phi(x, t_1) dx + \int_{t_1}^{t_2} \int_{\Omega} q_k(u) \phi_t dx dt + \int_0^T \int_{\partial\Omega} f \phi d\sigma dt + \int_{t_1}^{t_2} \int_{\cup_{i=1}^{i_0} \partial B_\delta(a_i)} g_i \phi d\sigma dt \end{aligned} \quad (2.22)$$

where

$$q_k(s) = \int_a^s p_k(\hat{s}) d\hat{s}$$

Since  $q_k(u) \rightarrow v - a$  as  $k \rightarrow \infty$ , letting  $k \rightarrow \infty$  in (2.22) we get (2.20). By a similar argument we get (2.21) and the lemma follows.  $\square$

**Lemma 2.6.** *Let  $0 < \delta < \delta_0$ ,  $f \in L^1(\partial\Omega \times (0, T))$ , and  $g_i \in L^1(\partial B_\delta(a_i) \times (0, T))$ ,  $i = 1, 2, \dots, i_0$ , be such that  $f \leq M$  for some constant  $M > 0$ . Suppose  $u \in C^{2,1}(\overline{\Omega}_\delta \times (0, T))$  is a solution of (1.4) in  $\Omega_\delta \times (0, T)$ . Let  $v = \max(u, 1)$ . Then there exists a constant  $C > 0$  such that*

$$\sup_{t_2 \leq t \leq t_0} \int_{D_r} v^{\alpha+1}(x, t) dx + \iint_S |\nabla v^{\frac{\alpha+m}{2}}|^2 dx dt \leq C\{(\rho - r)^{-2} + (t_2 - t_1)^{-1}\}(1 + \alpha^2) \iint_R v^{\alpha+1} dx dt \quad (2.23)$$

holds for any  $\alpha > 0$ ,  $0 < \rho < R_0/5$ , and  $0 < m < 1$ .

*Proof.* Since the proof is similar to that of Lemma 1.1 of [Hu1], we will only sketch the argument here. We first choose  $\eta \in C^\infty(\mathbb{R}^{n+1})$ ,  $0 \leq \eta \leq 1$ , such that  $\eta(x, s) = 0$  for all  $(x, s) \notin R$ ,  $\eta(x, s) = 1$  for all  $(x, s) \in S$ , and  $|\eta_t| \leq C(t_2 - t_1)^{-1}$ ,  $|\nabla \eta| \leq C(\rho - r)^{-1}$ , for some constant  $C > 0$ . By Lemma 2.5 the function  $v = \max(u, 1)$  satisfies (2.21). Since the function  $v^\alpha \eta^2 \in C(\overline{\Omega}_\delta \times (0, T))$  and its first order weak derivatives belong to  $L^2(\Omega_\delta \times (0, T))$ , putting  $\phi = v^\alpha \eta^2$  in (2.21) and simplifying

$$\begin{aligned} & \frac{1}{\alpha + 1} \int_{D_r} v^{\alpha+1}(x, t) dx + \iint_{R(t)} \nabla(v^\alpha \eta^2) \cdot \nabla(v^\alpha \eta^2) dx ds \\ & \leq \frac{2}{\alpha + 1} \iint_{R(t)} v^{\alpha+1} \eta \eta_t dx ds + M \iint_{(\partial D_\rho \cap \partial\Omega) \times (t_1, t)} v^\alpha \eta^2 d\sigma ds. \end{aligned} \quad (2.24)$$

Then

$$\begin{aligned} & \frac{1}{\alpha + 1} \int_{D_r} v^{\alpha+1}(x, t) dx + \frac{4\alpha m}{(\alpha + m)^2} \iint_{R(t)} \eta^2 |\nabla v^{\frac{\alpha+m}{2}}|^2 dx ds \\ & \leq \frac{2}{\alpha + 1} \iint_{R(t)} v^{\alpha+1} \eta \eta_t dx ds + 2m \iint_{R(t)} v^{\alpha+m-1} \eta |\nabla v| |\nabla \eta| dx ds + M \iint_{(\partial D_\rho \cap \partial\Omega) \times (t_1, t_0)} v^\alpha \eta^2 d\sigma ds \\ & \leq \frac{2}{\alpha + 1} \iint_{R(t)} v^{\alpha+1} \eta \eta_t dx ds + \frac{2m\alpha}{(\alpha + m)^2} \iint_{R(t)} \eta^2 |\nabla v^{\frac{\alpha+m}{2}}|^2 dx ds + \frac{2m}{\alpha} \iint_{R(t)} v^{\alpha+m} |\nabla \eta|^2 dx ds \\ & \quad + M \iint_{(\partial D_\rho \cap \partial\Omega) \times (t_1, t)} v^\alpha \eta^2 d\sigma ds. \end{aligned}$$

Hence

$$\frac{\alpha}{\alpha + 1} \int_{D_r} v^{\alpha+1}(x, t) dx + \frac{2m\alpha^2}{(\alpha + m)^2} \iint_{R(t)} \eta^2 |\nabla v^{\frac{\alpha+m}{2}}|^2 dx ds \leq 2 \iint_{R(t)} v^{\alpha+1} (|\eta_t| + |\nabla \eta|^2) dx ds + MI \quad (2.25)$$

where

$$I = \alpha \iint_{(\partial D_\rho \cap \partial\Omega) \times (t_1, t)} v^\alpha \eta^2 d\sigma ds.$$

By an argument similar to the proof of Lemma 1.1 of [Hu1],

$$\begin{aligned}
I &\leq \alpha \int_{t_1}^t \int_{\Omega} |\partial_{x_n}(v^\alpha \eta^2)| dx dt \\
&\leq \alpha^2 \iint_{R(t)} v^{\alpha-1} \eta^2 |\nabla v| dx ds + 2\alpha \iint_{R(t)} v^\alpha \eta |\nabla \eta| dx ds \\
&\leq \alpha^2 \iint_{R(t)} v^{\alpha+m-1} \eta^2 |\nabla v| dx ds + 2\alpha \iint_{R(t)} v^\alpha \eta |\nabla \eta| dx ds \\
&\leq \frac{2\alpha^2}{\alpha+m} \iint_{R(t)} \eta^2 v^{\frac{\alpha+m}{2}} |\nabla v^{\frac{\alpha+m}{2}}| dx ds + 2\alpha \iint_{R(t)} v^\alpha \eta |\nabla \eta| dx ds \\
&\leq \frac{m\alpha^2}{(\alpha+m)^2(M+1)} \iint_R \eta^2 |\nabla v^{\frac{\alpha+m}{2}}|^2 dx ds + \alpha^2 m^{-1}(M+1) \iint_R v^{\alpha+m} \eta^2 dx ds + 2\alpha \iint_{R(t)} v^\alpha \eta |\nabla \eta| dx ds.
\end{aligned} \tag{2.26}$$

By (2.25) and (2.26) we get (2.23) and the lemma follows.  $\square$

By Lemma 2.6, an argument similar to the proof in Section 2 of [Hs3], and a compactness argument we have the following result.

**Proposition 2.7.** *Let  $n \geq 3$ ,  $0 < m \leq \frac{n-2}{n}$ ,  $p > \frac{n(1-m)}{2}$ ,  $0 < \delta < \delta_0$ , and let  $f \in L^1(\partial\Omega \times (0, T))$  be such that  $f \leq M$  for some constant  $M > 0$ . Suppose  $u \in C^{2,1}(\overline{\Omega}_\delta \times (0, T))$  is a solution of (1.4) in  $\Omega_\delta \times (0, T)$ . Then for any  $0 < t_1 < t_2 < T$ ,  $r_1 \in (0, \delta_0/2)$ , there exists constants  $\theta > 0$  and  $C > 0$  depending on  $M$ ,  $m$ , and  $n$  such that*

$$\|u\|_{L^\infty(E_{r_1} \times (t_2, T))} \leq C \left\{ 1 + \int_{t_1}^T \int_{E_{2r_1}} u^p dx dt \right\}^{\frac{\theta}{p}}$$

where  $E_{r_1} = \{x \in \Omega : \text{dist}(x, \partial\Omega) < r_1\}$ .

Similarly we have the following result.

**Proposition 2.8.** *Let  $n \geq 3$ ,  $0 < m \leq \frac{n-2}{n}$ ,  $p > \frac{n(1-m)}{2}$ ,  $0 < \delta < \delta_0$ , and let  $f \in L^1(\partial\Omega \times (0, T))$  and  $g_i \in L^1(\partial B_\delta(a_i) \times (0, T))$  be such that  $f \leq M$  and  $g_i \leq M$  for all  $i = 1, \dots, i_0$  and some constant  $M > 0$ . Suppose  $u \in C^{2,1}(\overline{\Omega}_\delta \times (0, T))$  is a solution of (1.4) in  $\Omega_\delta \times (0, T)$ . Then for any  $0 < t_1 < t_2 < T$ , there exist constants  $\theta > 0$  and  $C > 0$  depending on  $M$ ,  $\delta$ ,  $m$ , and  $n$  such that*

$$\|u\|_{L^\infty(\Omega_\delta \times (t_2, T))} \leq C \left( 1 + \iint_{\Omega_\delta \times (t_1, T]} u^p dx dt \right)^{\frac{\theta}{p}}.$$

**Proposition 2.9.** *Let  $n \geq 3$ ,  $0 < m \leq \frac{n-2}{n}$ ,  $p > \frac{n(1-m)}{2}$ ,  $0 < \delta < \delta_0$ , and let  $f \in L^1(\partial\Omega \times (0, T))$  and  $g_i \in L^1(\partial B_\delta(a_i) \times (0, T))$  be such that  $f \leq M$  and  $g_i \leq M$  for all  $i = 1, \dots, i_0$  and some constant  $M > 0$ . Suppose  $0 \leq u_0 \in L^p(\Omega_\delta)$  and  $u \in C^{2,1}(\overline{\Omega}_\delta \times (0, T))$  is a solution of (1.4) in  $\Omega_\delta \times (0, T)$ . Then for any  $0 < t_1 < T$  there exist constants  $\theta > 0$  and  $C > 0$  depending on  $M$ ,  $\delta$ ,  $m$ , and  $n$  such that*

$$\|u\|_{L^\infty(\Omega_\delta \times (t_1, T))} \leq C \left( 1 + \int_{\Omega_\delta} u_0^p dx \right)^{\frac{\theta}{p}}.$$

*Proof.* Let  $0 < a \leq 1$  and  $v = \max(u, a)$ . As before by Lemma 2.5 the function  $v$  satisfies (2.21). Since the function  $v^{p-1} \in C(\overline{\Omega}_\delta \times (0, T))$  and its first order weak derivatives belong to  $L^2(\Omega_\delta \times (0, T))$ , by

(2.21),

$$\begin{aligned}
\frac{\partial}{\partial t} \left( \int_{\Omega_\delta} v^p dx \right) &= p \int_{\Omega_\delta} v^{p-1} v_t dx \\
&\leq -p \int_{\Omega_\delta} \nabla v^{p-1} \cdot \nabla v^m dx + p \int_{\partial\Omega_\delta \cup (\cup_{i=1}^{i_0} \partial B_\delta(a_i))} v^{p-1} \frac{\partial v^m}{\partial \nu} d\sigma \\
&\leq -p(p-1)m \int_{\Omega_\delta} v^{p+m-3} |\nabla v|^2 dx + Mp \int_{\partial\Omega_\delta \cup (\cup_{i=1}^{i_0} \partial B_\delta(a_i))} v^{p-1} d\sigma. \tag{2.27}
\end{aligned}$$

By an argument similar to the proof of Lemma 1.1 of [Hu1] there exists a constant  $C_1 > 0$  such that,

$$\begin{aligned}
\int_{\partial\Omega_\delta \cup (\cup_{i=1}^{i_0} \partial B_\delta(a_i))} v^{p-1} d\sigma &\leq C_1 \left( \int_{\Omega_\delta} |\nabla v^{p-1}| dx + \int_{\Omega_\delta} v^{p-1} dx \right) \\
&= (p-1)C_1 \int_{\Omega_\delta} v^{p-2} |\nabla v| dx + C_1 \int_{\Omega_\delta} v^{p-1} dx \\
&\leq \frac{m(p-1)}{2M} \int_{\Omega_\delta} v^{p+m-3} |\nabla v|^2 dx + \frac{(p-1)MC_1^2}{2m} \int_{\Omega_\delta} v^{p-m-1} dx + C_1 \int_{\Omega_\delta} v^{p-1} dx. \tag{2.28}
\end{aligned}$$

By (2.27) and (2.28),

$$\begin{aligned}
\frac{\partial}{\partial t} \left( \int_{\Omega_\delta} v^p dx \right) &\leq C_2 \left( \int_{\Omega_\delta} v^{p-m-1} dx + \int_{\Omega_\delta} v^{p-1} dx \right) \\
&\leq \frac{2C_2}{a^m} \int_{\Omega_\delta} v^{p-1} dx \\
&\leq \frac{2C_2 |\Omega|^{1/p}}{a^m} \left( \int_{\Omega_\delta} v^p dx \right)^{\frac{p-1}{p}}
\end{aligned}$$

for some constant  $C_2 > 0$ . Hence

$$\left( \int_{\Omega_\delta} u^p(x, t) dx \right)^{\frac{1}{p}} \leq \left( \int_{\Omega_\delta} v^p(x, t) dx \right)^{\frac{1}{p}} \leq \left( \int_{\Omega_\delta} (u_0 + a)^p dx \right)^{\frac{1}{p}} + 2C_2 p^{-1} a^{-m} |\Omega|^{1/p} t \quad \forall 0 < t < T. \tag{2.29}$$

By (2.29) and Proposition 2.8, Proposition 2.9 follows.  $\square$

**Lemma 2.10.** *Let  $n \geq 3$ ,  $0 < m \leq \frac{n-2}{n}$ ,  $p > \frac{n(1-m)}{2}$ ,  $0 < \delta < \delta_2 < \delta_1 < \delta_0$ , and  $g_i \in L^1(\partial B_\delta(a_i) \times (0, T))$  for all  $i = 1, \dots, i_0$  and let  $f \in L^1(\partial\Omega \times (0, T))$  be such that  $f \leq M$  for some constant  $M > 0$ . Suppose  $0 \leq u_0 \in L^p(\Omega_\delta)$  and  $u \in C^{2,1}(\overline{\Omega}_\delta \times (0, T))$  is a solution of (1.4) in  $\Omega_\delta \times (0, T)$ . Then there exists a constant  $C > 0$  depending on  $M, \delta_1, \delta_2, m$ , and  $n$  such that*

$$\left( \int_{E_{\delta_0-\delta_1}} u^p(x, t) dx \right)^{\frac{1-m}{p}} \leq \left( \int_{E_{\delta_0-\delta_3}} (u_0 + 1)^p dx \right)^{\frac{1-m}{p}} + Ct \quad \forall 0 < t < T$$

holds for any  $0 < \delta \leq \delta_2$  where  $\delta_3 = (\delta_1 + \delta_2)/2$  and  $E_r = \{x \in \Omega : \text{dist}(x, \partial\Omega) < r\}$  for any  $0 < r < \delta_0$ .

*Proof.* We choose  $\phi \in C^\infty(\overline{\Omega})$ ,  $0 \leq \phi \leq 1$ , such that  $\phi(x) = 1$  for all  $x \in E_{\delta_0-\delta_1}$ ,  $\phi(x) = 0$  for all  $x \notin E_{\delta_0-\delta_3}$ . Let  $\eta = \phi^\alpha$  for some constant  $\alpha > 0$  to be chosen later and let  $v = \max(u, 1)$ . By Lemma

2.5 the function  $v$  satisfies (2.21). Since the function  $v^{p-1}\eta^2 \in C(\overline{\Omega_\delta} \times (0, T))$  and its first order weak derivatives belong to  $L^2(\Omega_\delta \times (0, T))$ , by (2.21),

$$\begin{aligned} \frac{\partial}{\partial t} \left( \int_{\Omega_\delta} v^p \eta^2 dx \right) &= p \int_{\Omega_\delta} v^{p-1} \eta^2 v_t dx \\ &\leq -p \int_{\Omega_\delta} \nabla(v^{p-1} \eta^2) \cdot \nabla v^m dx + p \int_{\partial\Omega} v^{p-1} \eta^2 \frac{\partial v^m}{\partial \nu} d\sigma \\ &\leq -p(p-1)m \int_{\Omega_\delta} v^{p+m-3} \eta^2 |\nabla v|^2 dx + 2pm \int_{\Omega_\delta} v^{p+m-2} \eta |\nabla \eta| |\nabla v| dx \\ &\quad + pM \int_{\partial\Omega} v^{p-1} \eta^2 d\sigma. \end{aligned} \quad (2.30)$$

Now

$$2 \int_{\Omega_\delta} v^{p+m-2} \eta |\nabla \eta| |\nabla v| dx \leq \frac{(p-1)}{4} \int_{\Omega_\delta} v^{p+m-3} \eta^2 |\nabla v|^2 dx + \frac{4}{p-1} \int_{\Omega_\delta} v^{p+m-1} |\nabla \eta|^2 dx. \quad (2.31)$$

By an argument similar to the proof of Lemma 1.1 of [Hu1] there exists a constant  $C_1 > 0$  such that,

$$\begin{aligned} &\int_{\partial\Omega} v^{p-1} \eta^2 d\sigma \\ &\leq C_1 \int_{\Omega_\delta} |\nabla(v^{p-1} \eta^2)| dx \\ &\leq (p-1)C_1 \int_{\Omega_\delta} v^{p-2} \eta^2 |\nabla v| dx + 2C_1 \int_{\Omega_\delta} v^{p-1} \eta |\nabla \eta| dx \\ &\leq \frac{m(p-1)}{2M} \int_{\Omega_\delta} v^{p+m-3} \eta^2 |\nabla v|^2 dx + \frac{(p-1)MC_1^2}{2m} \int_{\Omega_\delta} v^{p-m-1} \eta^2 dx + 2C_1 \int_{\Omega_\delta} v^{p-1} \eta |\nabla \eta| dx. \end{aligned} \quad (2.32)$$

By (2.30), (2.31) and (2.32),

$$\begin{aligned} &\frac{\partial}{\partial t} \left( \int_{\Omega_\delta} v^p \eta^2 dx \right) \\ &\leq C_2 \left( \int_{\Omega_\delta} v^{p-m-1} \eta^2 dx + \int_{\Omega_\delta} v^{p-1} \eta |\nabla \eta| dx + \int_{\Omega_\delta} v^{p+m-1} |\nabla \eta|^2 dx \right) \\ &\leq C_2 \left[ \int_{\Omega_\delta} v^{p-1} \eta^2 dx + \int_{\Omega_\delta} (v^p \eta^2)^{\frac{p-1}{p}} |\nabla \eta| \eta^{-1+\frac{2}{p}} dx + \int_{\Omega_\delta} (v^p \eta^2)^{\frac{p+m-1}{p}} (|\nabla \eta| \eta^{-1+\frac{1-m}{p}})^2 dx \right] \end{aligned} \quad (2.33)$$

for some constant  $C_2 > 0$ . Now

$$|\nabla \eta| \eta^{-1+\frac{2}{p}} = \alpha \phi^{\frac{2\alpha}{p}-1} |\nabla \phi| \in L^\infty \quad \text{if } \alpha > \frac{p}{2} \quad (2.34)$$

and

$$|\nabla \eta| \eta^{-1+\frac{1-m}{p}} = \alpha \phi^{\frac{(1-m)}{p}\alpha-1} |\nabla \phi| \in L^\infty \quad \text{if } \alpha > \frac{p}{1-m}. \quad (2.35)$$

We now choose  $\alpha > \max\left(\frac{p}{2}, \frac{p}{1-m}\right)$ . Since

$$\int_{\Omega_\delta} v^p \eta^2 dx \geq \int_{\Omega} \eta^2 dx > 0, \quad (2.36)$$

by (2.33), (2.34) and (2.35),

$$\begin{aligned}
\frac{\partial}{\partial t} \left( \int_{\Omega_\delta} v^p \eta^2 dx \right) &\leq C_3 \left[ \int_{\Omega_\delta} v^{p-1} \eta^2 dx + \int_{\Omega_\delta} (v^p \eta^2)^{\frac{p-1}{p}} dx + \int_{\Omega_\delta} (v^p \eta^2)^{\frac{p+m-1}{p}} dx \right] \\
&\leq C_4 \left[ \left( \int_{\Omega_\delta} v^p \eta^2 dx \right)^{\frac{p-1}{p}} + \left( \int_{\Omega_\delta} v^p \eta^2 dx \right)^{\frac{p+m-1}{p}} \right] \\
&\leq C_5 \left( \int_{\Omega_\delta} v^p \eta^2 dx \right)^{\frac{p+m-1}{p}}
\end{aligned} \tag{2.37}$$

for some constants  $C_3 > 0$ ,  $C_4 > 0$ ,  $C_5 > 0$ . Integrating (2.37),

$$\left( \int_{\Omega_\delta} u^p(x, t) \eta^2(x) dx \right)^{\frac{1-m}{p}} \leq \left( \int_{\Omega_\delta} v^p(x, t) \eta^2(x) dx \right)^{\frac{1-m}{p}} \leq \left( \int_{\Omega_\delta} (u_0 + 1)^p \eta^2 dx \right)^{\frac{1-m}{p}} + C_6 t \quad \forall 0 < t < T$$

for some constant  $C_6 > 0$  and the lemma follows.  $\square$

By a similar argument we also have the following result.

**Proposition 2.11.** *Let  $n \geq 3$ ,  $0 < m \leq \frac{n-2}{n}$ ,  $p > \frac{n(1-m)}{2}$ ,  $0 < \delta < \delta_0$ . Let  $0 \leq u_0 \in L^p(\Omega_\delta)$ ,  $g_i \in L^1(\partial B_\delta(a_i) \times (0, T))$  for all  $i = 1, 2, \dots, i_0$ ,  $f \in L^1(\partial \Omega \times (0, T))$  be such that  $f \leq M$  for some constant  $M > 0$ . Suppose  $u \in C^{2,1}(\overline{\Omega_\delta} \times (0, T))$  is a solution of (1.4) in  $\Omega_\delta \times (0, T)$ . Then for any  $0 < t_1 < T$ ,  $\delta < \delta_1 < \delta_2 \leq \delta_0$ , there exist constants  $\theta > 0$  and  $C > 0$  depending on  $M, m, n, t_1, \delta_1$ , and  $\delta_2$  such that*

$$\|u\|_{L^\infty(\Omega_{\delta_2} \times (t_1, T))} \leq C \left\{ 1 + \int_{\Omega_{\delta_1}} u_0^p dx \right\}^{\frac{\theta}{p}}.$$

and for any  $0 < t_1 < T$ ,  $R_2 > R_1 > 0$  such that  $B_{R_2}(x_1) \subset \Omega_\delta$ , there exist constants  $\theta > 0$  and  $C > 0$  depending on  $m, n, t_1, R_1$  and  $R_2$ , but independent of  $M$  such that

$$\|u\|_{L^\infty(B_{R_1}(x_1) \times (t_1, T))} \leq C \left\{ 1 + \int_{B_{R_2}(x_1)} u_0^p dx \right\}^{\frac{\theta}{p}}.$$

### 3. EXISTENCE OF SOLUTIONS FOR THE NEUMANN PROBLEM

In this section we will prove the existence of solutions of the Neumann problem (1.4). We first observe that by an argument similar to the proof of Proposition A.1 of [BV1] we have the following lemma.

**Lemma 3.1.** *Let  $n \geq 1$ ,  $0 < m < 1$ , and  $0 \leq v_0(x) \in L^1(B_{5R}(x_0))$ ,  $v_0 \not\equiv 0$ , such that  $\text{supp } v_0 \subset B_R(x_0) \subset \mathbb{R}^n$ . Let  $v$  be a weak solution of*

$$\begin{cases} v_t = \Delta v^m & \text{in } B_{5R}(x_0) \times (0, T_{v_0}) \\ v(x, t) = 0 & \text{on } \partial B_{5R}(x_0) \times (0, T_{v_0}) \\ v(x, 0) = v_0(x) & \text{in } B_{5R}(x_0) \end{cases} \tag{3.1}$$

where  $T_{v_0} > 0$  is the extinction time of  $v$ , then

$$v(x_1, t) \geq v(x_2, t) \quad \forall |x_1| \leq R, 4R \leq |x_2| \leq 5R.$$

By Lemma 3.1 and an argument similar to the proof in section 1 of [BV2] we have the following result.

**Lemma 3.2.** (cf. (1.18) and (1.27) of [BV2]) Let  $n, m$ , and  $v_0, v$ , and  $T_{v_0}$ , be as in Lemma 3.1. Then there exist constants  $C_1 > 0, C_2 > 0$ , such that

$$T_{v_0} \geq C_1 R^{2-n(1-m)} \left( \int_{B_R(x_0)} v_0 dx \right)^{1-m}$$

and

$$v^m(x, t) \geq C_2 R^{2-n} \|v_0\|_{L^1(B_R(x_0))} T_{v_0}^{-\frac{1}{1-m}} t^{\frac{m}{1-m}} \quad \forall |x| \leq R, t \in (0, t_*]$$

where

$$t_* = \frac{C_1}{2} R^{2-n(1-m)} \left( \int_{B_R(x_0)} v_0 dx \right)^{1-m}.$$

**Lemma 3.3.** Let  $n \geq 3, 0 < m \leq \frac{n-2}{n}$  and  $T > t_0 > 0$ . Suppose  $0 \leq u \in C(\Omega \times (t_0, T])$  satisfies (1.1) in  $\mathcal{D}(\Omega \times (t_0, T))$  and

$$\int_{\Omega} u(x, T) dx > 0. \quad (3.2)$$

Then

$$u(x, T) > 0 \quad \forall x \in \Omega.$$

*Proof.* Let

$$D(T) = \{x \in \Omega : u(x, T) > 0\}.$$

Since by (3.2) there exists a point  $x(T) \in \Omega$  such that  $u(x(T), T) > 0$ ,  $D(T) \neq \emptyset$ . Suppose that  $D(T) \neq \Omega$ . Then there exist a point  $x_0 \in \Omega \cap \partial D(T)$  such that

$$u(x_0, T) = 0. \quad (3.3)$$

We choose a constant  $R > 0$  such that  $B_{5R}(x_0) \subset \Omega$ . We claim that

$$\int_{B_{R/2}(x_0)} u(x, T) dx > 0. \quad (3.4)$$

Suppose not. Then

$$\int_{B_{R/2}(x_0)} u(x, T) dx = 0 \quad \Rightarrow \quad u(x, T) = 0 \quad \forall |x - x_0| \leq \frac{R}{2}$$

which contradicts the fact that  $x_0 \in \partial D(T)$ . Hence (3.4) holds. Let  $\psi \in C_0^\infty(B_{5R}(x_0))$ ,  $0 \leq \psi \leq 1$ , be such that  $\psi(x) = 1$  for all  $x \in B_{R/2}(x_0)$  and  $\psi(x) = 0$  for all  $x \in B_{5R}(x_0) \setminus B_R(x_0)$ . By the proof of Lemma 3.1 of [HP] there exist constants  $\alpha > 1$  and  $C_R > 0$  such that

$$\begin{aligned} & \left| \frac{\partial}{\partial t} \left( \int_{B_{5R}(x_0)} u \psi^\alpha dx \right) \right| \leq C_R \left( \int_{B_{5R}(x_0)} u \psi^\alpha dx \right)^m \quad \forall 0 < t < T \\ \Rightarrow & \left| \left( \int_{B_{5R}(x_0)} u(x, T) \psi^\alpha(x) dx \right)^{1-m} - \left( \int_{B_{5R}(x_0)} u(x, s) \psi^\alpha(x) dx \right)^{1-m} \right| \leq (1-m) C_R (T-s) \quad \forall T > s > 0. \end{aligned} \quad (3.5)$$

Let

$$\varepsilon_1 = \frac{1 - 2^{m-1}}{(1-m)C_R} \left( \int_{B_{5R}(x_0)} u(x, T) \psi^\alpha(x) dx \right)^{1-m}.$$

By (3.5) for any  $0 < T - s \leq \varepsilon_1$

$$\begin{aligned}
\int_{B_R(x_0)} u(x, s) dx &\geq \int_{B_{5R}(x_0)} u(x, s) \psi^\alpha(x) dx \\
&\geq \left[ \left( \int_{B_{5R}(x_0)} u(x, T) \psi^\alpha(x) dx \right)^{1-m} - (1-m)C_R(T-s) \right]^{\frac{1}{1-m}} \\
&\geq \frac{1}{2} \int_{B_{5R}(x_0)} u(x, T) \psi^\alpha(x) dx \\
&\geq \frac{1}{2} \int_{B_{R/2}(x_0)} u(x, T) dx.
\end{aligned} \tag{3.6}$$

Let  $v_{0,\tau}(x) = u(x, T - \tau)\chi_{B_R(x_0)}$ . By the discussion on P.537 of [BV2] there exists a unique weak minimal solution  $v^\tau$  of (3.1) with initial value  $v_{0,\tau}$ . Let  $T_{v_{0,\tau}}$  be the extinction time of  $v^\tau$ . Then by (3.4), (3.6) and Lemma 3.2,

$$T_{v_{0,\tau}} \geq c_0 > 0 \quad \forall 0 < \tau \leq \varepsilon_1.$$

where

$$c_0 = C_1 R^{2-n(1-m)} \left( \frac{1}{2} \int_{B_{R/2}(x_0)} u(x, T) dx \right)^{1-m} \quad \text{and} \quad C_1 > 0 \text{ is as in Lemma 3.2.}$$

Let  $\tau_1 = \min(c_0, \varepsilon_1)/2$ . By Lemma 3.2 there exists a constant  $c_1 > 0$  such that

$$v^{\tau_1}(x, \tau_1) \geq c_1 \quad \forall |x| \leq R. \tag{3.7}$$

Since  $v^{\tau_1}$  is the unique weak minimal solution of (3.1) with initial value  $v_{0,\tau_1}$ , by the maximum principle,

$$u^m(x, T) \geq (v^{\tau_1})^m(x, \tau_1) > 0 \quad \forall |x| \leq R. \tag{3.8}$$

By (3.7) and (3.8) we get  $u^m(x_0, T) \geq c_1 > 0$ . This contradicts (3.3). Hence  $D(T) = \Omega$  and the lemma follows.  $\square$

**Lemma 3.4.** *Let  $n \geq 1$ ,  $0 < m < 1$ ,  $0 < \delta < \delta_0$ ,  $0 < u_0 \in C^2(\overline{\Omega_\delta})$ . Let  $f \in C^\infty(\partial\Omega \times [0, T])$  and  $g_i \in C^\infty(\partial B_\delta(a_i) \times [0, T])$  for all  $i = 1, \dots, i_0$  be such that  $f, g_i$ , are nonnegative monotone decreasing functions of  $t \in (0, T)$ , and  $f \leq M$ ,  $g_i \leq M$ , for all  $i = 1, \dots, i_0$  and some constant  $M > 0$ . Suppose  $u \in C^{2,1}(\overline{\Omega_\delta} \times [0, T])$  is a positive solution of (1.4). Then  $u$  satisfies (1.6) in  $\Omega_\delta \times (0, T)$ .*

*Proof.* Let

$$q = \frac{u_t}{u}, \quad \varepsilon = \min_{\overline{\Omega_\delta}} u_0, \quad a = \frac{\varepsilon}{(1-m)(\|\Delta u_0^m\|_{L^\infty(\Omega_\delta)} + 1)}$$

and

$$\bar{q} = q - \frac{1}{(1-m)(a+t)}.$$

By direct computation,

$$q_t = mu^{m-1}\Delta q + 2mu^{m-2}\nabla u \cdot \nabla q + (m-1)q^2 \quad \text{in } \Omega_\delta \times (0, T). \tag{3.9}$$

Then  $\bar{q}$  satisfies

$$\begin{cases} \bar{q}_t = mu^{m-1}\Delta \bar{q} + 2mu^{m-2}\nabla u \cdot \nabla \bar{q} - (1-m)\bar{q} \left( \bar{q} + \frac{2}{(1-m)(a+t)} \right) & \text{in } \Omega_\delta \times (0, T) \\ \bar{q}(x, 0) \leq 0 & \text{on } \Omega_\delta. \end{cases} \tag{3.10}$$



Since

$$\begin{aligned} f_t &= \frac{\partial}{\partial t} \left( \frac{\partial u^m}{\partial \nu} \right) = mu^{m-1} u_{\nu t} - m(1-m)u^{m-2} u_{\nu} u_t \quad \forall (x, t) \in \partial\Omega \times (0, T) \\ \Rightarrow \quad u_{\nu t} &= \frac{f_t u^{1-m}}{m} + (1-m) \frac{u_{\nu} u_t}{u} \quad \forall (x, t) \in \partial\Omega \times (0, T), \end{aligned} \quad (3.11)$$

we have

$$\begin{aligned} \frac{\partial q}{\partial \nu} &= \frac{u_{\nu t}}{u} - \frac{u_{\nu} u_t}{u^2} = \frac{f_t}{mu^m} - \frac{qf}{u^m} \quad \text{on } \partial\Omega \times (0, T) \\ &= \frac{f_t}{mu^m} - \frac{\bar{q}f}{u^m} - \frac{f}{(1-m)(a+t)u^m} \quad \text{on } \partial\Omega \times (0, T). \end{aligned} \quad (3.12)$$

Similarly

$$\frac{\partial q}{\partial \nu} = \frac{g_{i,t}}{mu^m} - \frac{\bar{q}g_i}{u^m} - \frac{g_i}{(1-m)(a+t)u^m} \quad \text{on } \partial B_{\delta}(a_i) \times (0, T) \quad \forall i = 1, \dots, i_0.$$

Let  $0 < T_1 < T$  and suppose  $\bar{q}$  attains a positive maximum on  $\bar{\Omega}_{\delta} \times (0, T_1]$  at  $(x_0, t_0)$  for some  $x_0 \in \bar{\Omega}_{\delta}$ ,  $0 \leq t_0 \leq T_1$ . Suppose  $x_0 \in \Omega_{\delta}$  and  $t_0 > 0$ . Then

$$\bar{q}_t \geq 0, \quad \nabla \bar{q} = 0 \quad \text{and} \quad \Delta \bar{q} \leq 0 \quad \text{at } (x_0, t_0). \quad (3.13)$$

Hence by (3.10) and (3.13),

$$\begin{aligned} 0 \leq \bar{q}_t &= mu^{m-1} \Delta \bar{q} + 2mu^{m-2} \nabla u \cdot \nabla \bar{q} - (1-m)\bar{q} \left( \bar{q} + \frac{2}{(1+m)(a+t)} \right) \\ &\leq - (1-m)\bar{q} \left( \bar{q} + \frac{2}{(1+m)(a+t)} \right) < 0 \quad \text{at } (x_0, t_0). \end{aligned}$$

Contradiction arises. Hence either  $x_0 \in \partial\Omega_{\delta}$  or  $t_0 = 0$ . Suppose  $x_0 \in \partial\Omega_{\delta}$  and  $t_0 > 0$ . Without loss of generality we may assume that  $x_0 \in \partial\Omega$ . By the strong maximum principle,

$$\frac{\partial \bar{q}}{\partial \nu}(x_0, t_0) > 0. \quad (3.14)$$

Then by (3.12) and (3.14),

$$\begin{aligned} 0 &< \frac{f_t}{mu^m} - \frac{\bar{q}f}{u^m} - \frac{f}{(1-m)(a+t)u^m} \leq -\frac{\bar{q}f}{u^m} \quad \text{at } (x_0, t_0) \\ \Rightarrow \quad \bar{q}f &< 0 \quad \text{at } (x_0, t_0) \end{aligned}$$

Since  $\bar{q}(x_0, t_0) > 0$  and  $f(x_0, t_0) \geq 0$ , contradiction arises. Hence  $t_0 = 0$ . Since  $\bar{q}(x, 0) \leq 0$  on  $\Omega_{\delta}$ ,

$$\begin{aligned} \bar{q}(x, t) &\leq 0 \quad \forall x \in \Omega_{\delta}, 0 < t < T_1 \\ \Rightarrow \quad \bar{q}(x, t) &\leq 0 \quad \forall x \in \Omega_{\delta}, 0 < t < T \quad \text{as } T_1 \rightarrow T \\ \Rightarrow \quad u_t &\leq \frac{u}{(1-m)(a+t)} \leq \frac{u}{(1-m)t} \quad \text{in } \Omega_{\delta} \times (0, T) \end{aligned}$$

and the lemma follows.  $\square$

We are now ready for the proof of Theorem 1.1.

*Proof of Theorem 1.1.* By Lemma 2.1 the solution of (1.4) is unique. Hence it remains to prove the existence of solution of (1.4). We will use a modification of the proof of Lemma 2.1 of [Hs2] to prove this theorem. We divide the proof into three cases.

**Case 1:**  $0 < u_0 \in C^\infty(\overline{\Omega_\delta})$ ,  $f \in C^\infty(\partial\Omega \times [0, \infty))$ ,  $g_i \in C^\infty(\partial B_\delta(a_i) \times [0, \infty))$ , such that  $g_i(x, t) = \alpha_i$  for all  $(x, t) \in \partial B_\delta(a_i) \times [0, \delta')$ ,  $i = 1, \dots, i_0$ , and  $f(x, t) = 0$  for all  $(x, t) \in \partial\Omega \times [0, \delta')$  for some constants  $\delta' > 0$  and  $\alpha_1, \dots, \alpha_{i_0} \in \mathbb{R}^+$ , respectively.

Let  $\varepsilon_0 = \inf_{\overline{\Omega_\delta}} u_0$ . Then  $\varepsilon_0 > 0$ . We first choose a sequence of functions  $\{\phi_j\}_{j=1}^\infty \subset C^\infty(\Omega)$ ,  $0 \leq \phi_j \leq 1$  for all  $j \in \mathbb{Z}^+$ , such that

$$\phi_j(x) = \begin{cases} 1 & \text{if } \mathbf{dist}(x, \partial\Omega) \leq \frac{\delta_0 - \delta}{3(j+1)} \\ 0 & \text{if } \mathbf{dist}(x, \partial\Omega) > \frac{\delta_0 - \delta}{3j} \end{cases}$$

for all  $j \in \mathbb{Z}^+$  and a sequence of functions  $\{\psi_j\}_{j=1}^\infty \subset C_0^\infty(\mathbb{R}^N)$ ,  $0 \leq \psi_j \leq 1$  for all  $j \in \mathbb{Z}^+$ , such that

$$\psi_j(x) = \begin{cases} 1 & \text{if } |x| \leq \delta + \frac{\delta_0 - \delta}{3(j+1)} \\ 0 & \text{if } |x| \geq \delta + \frac{\delta_0 - \delta}{3j} \end{cases}$$

for all  $j \in \mathbb{Z}^+$ . Then  $0 \leq \phi_{j+1} \leq \phi_j \leq 1$  and  $0 \leq \psi_{j+1} \leq \psi_j \leq 1$  for any  $j \in \mathbb{Z}^+$ . For any  $j \in \mathbb{Z}^+$  and  $x \in \Omega$  let

$$u_{0,j}(x) = u_0(x) \left( 1 - \phi_j(x) - \sum_{i=1}^{i_0} \psi_j(x - a_i) \right) + \delta^{\frac{2}{m}} \sum_{i=1}^{i_0} \frac{\alpha_i^{\frac{1}{m}}}{|x - a_i|^{\frac{1}{m}}} \cdot \psi_j(x - a_i) + \varepsilon_0 \phi_j$$

and

$$\varepsilon_1 = \min \left\{ \varepsilon_0, \left( \frac{\delta^2 \alpha}{\mathbf{diam} \Omega} \right)^{\frac{1}{m}} \right\}, \quad \text{where } \alpha = \min \{ \alpha_1, \dots, \alpha_{i_0} \}.$$

Then

$$\varepsilon_1 \leq u_{0,j} \leq u_0 + C_1 \quad \text{in } \Omega_\delta \quad \forall j \in \mathbb{Z}^+ \quad \text{and} \quad \|u_{0,j} - u_0\|_{L^1(\Omega_\delta)} \rightarrow 0 \quad \text{as } j \rightarrow \infty \quad (3.15)$$

for some constant  $C_1 > 0$  depending on  $m, \alpha_1, \dots, \alpha_{i_0}, \delta$  and

$$\begin{cases} \frac{\partial u_{0,j}^m}{\partial \nu} = 0 & \text{on } \partial\Omega \quad \forall j \in \mathbb{Z}^+ \\ \frac{\partial u_{0,j}^m}{\partial \nu} = \alpha_i & \text{on } \partial B_\delta(a_i) \quad \forall j \in \mathbb{Z}^+, i = 1, 2, \dots, i_0. \end{cases}$$

We will now use an argument similar to the proof of Theorem 3.5 of [Hu1] to prove the existence of a solution of (1.4) in  $\Omega_\delta \times (0, \infty)$  with initial value  $u_{0,j}$ . Let  $n_1 = \varepsilon_1^m$  and  $n_2 = (\|u_0\|_{L^\infty} + C_1)^m$ . We choose a monotone decreasing function  $H \in C^\infty(\mathbb{R})$ ,  $H > 0$ , such that  $H(s) = ms^{1-\frac{1}{m}}$  for  $n_1/2 \leq s \leq 2n_2$ ,  $H(s) = m(n_1/4)^{1-\frac{1}{m}}$  for  $s \leq n_1/4$ ,  $H(s) = m(4n_1/4)^{1-\frac{1}{m}}$  for  $s \geq 4n_2$ . Then by standard theory for non-degenerate parabolic equation [LSU] such that the problem

$$\begin{cases} v_t = H(v) \Delta v & \text{in } \Omega_\delta \times (0, \infty) \\ \frac{\partial v}{\partial \nu} = f & \text{on } \partial\Omega \times (0, \infty) \\ \frac{\partial v}{\partial \nu} = g_i & \text{on } \partial B_\delta(a_i) \times (0, \infty) \quad \forall i = 1, \dots, i_0 \\ v(x, 0) = u_{0,j}(x, 0)^m & \text{in } \Omega_\delta \end{cases}$$

has a classical solution  $v_j \in C^{2,1}(\overline{\Omega}_\delta \times (0, \infty))$ . By the maximum principle for non-degenerate parabolic equation [LSU],  $n_1 \leq v_j \leq n_2$  on  $\overline{\Omega}_\delta \times (0, \infty)$ . Hence  $H(v_j) = mv_j^{1-\frac{1}{m}}$  and thus the function  $u_j = v_j^{\frac{1}{m}} \in C^{2,1}(\overline{\Omega}_\delta \times (0, \infty))$  is a solution of (1.4) in  $\Omega_\delta \times (0, \infty)$  with initial value  $u_{0,j}$  such that

$$u_j \geq \varepsilon_1 \quad \text{in } \overline{\Omega}_\delta \times [0, \infty). \quad (3.16)$$

Since

$$\frac{\partial}{\partial t} \left( \int_{\Omega_\delta} u_j dx \right) = \int_{\Omega_\delta} \Delta u_j^m dx = \int_{\partial\Omega} f d\sigma + \sum_{i=1}^{i_0} \int_{\partial B_\delta(a_i)} g_i d\sigma \quad \forall t > 0, j \in \mathbb{Z}^+,$$

integrating over  $t$  we have

$$\int_{\Omega_\delta} u_j(x, t) dx = \int_{\Omega_\delta} u_{0,j} dx + \int_0^t \int_{\partial\Omega} f d\sigma ds + \sum_{i=1}^{i_0} \int_0^t \int_{\partial B_\delta(a_i)} g_i d\sigma ds \quad t > 0. \quad (3.17)$$

We will now show that  $u_j$  converges to a solution  $u$  of (1.4) as  $j \rightarrow \infty$ . Let  $t_2 > t_1 > 0$ . Then by (3.15) and Proposition 2.9 there exists a constant  $C_2 > 0$  such that

$$u_j \leq C_2 \quad \forall x \in \overline{\Omega}_\delta, t_1 \leq t \leq t_2, j \in \mathbb{Z}^+. \quad (3.18)$$

Hence by (3.16) and (3.18) the equation (1.1) for the sequence  $\{u_j\}_{j=1}^\infty$  is uniformly parabolic on  $\overline{\Omega}_\delta \times [t_1, t_2]$  for any  $t_2 > t_1 > 0$ . Thus by the Schauder estimates [LSU] (Theorem 3.1 and Theorem 5.4 in chapter 5 of [LSU]), for any  $t_2 > t_1 > 0$ ,

$$\sup_{\overline{\Omega}_\delta \times [t_1, t_2]} (|\nabla u_j| + |\partial_{x_k x_l}^2 u_j| + |u_{j,t}|) \leq C_3 \quad \forall k, l = 1, \dots, n, j \in \mathbb{Z}^+$$

and

$$\sup_{\substack{y, y' \in \overline{\Omega}_\delta \\ s, s' \in [t_1, t_2]}} \frac{|\partial_{x_k x_l}^2 u_j(y, s) - \partial_{x_k x_l}^2 u_j(y', s')|}{|y - y'|^\alpha} + \sup_{\substack{y, y' \in \overline{\Omega}_\delta \\ s, s' \in [t_1, t_2]}} \frac{|u_{j,t}(y, s) - u_{j,t}(y', s')|}{|s - s'|^\frac{\alpha}{2}} \leq C_4 \quad \forall k, l = 1, \dots, n, j \in \mathbb{Z}^+$$

for some constants  $C_3 > 0$ ,  $C_4 > 0$ ,  $0 < \alpha < 1$ . Hence the sequence  $\{u_j\}_{j=1}^\infty$  is uniformly Hölder continuous in  $C^{2,1}(\overline{\Omega}_\delta \times [t_1, t_2])$  for any  $t_2 > t_1 > 0$ . By the Ascoli Theorem and a diagonalization argument the sequence  $\{u_j\}_{j=1}^\infty$  has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly on every compact subset of  $\overline{\Omega}_\delta \times (0, \infty)$  to a solution  $u \in C^{2,1}(\overline{\Omega}_\delta \times (0, \infty))$  of (1.1) in  $\overline{\Omega}_\delta \times (0, \infty)$  which satisfies

$$\frac{\partial u^m}{\partial \nu} = f \quad \text{on } \partial\Omega \times (0, \infty) \quad \text{and} \quad \frac{\partial u^m}{\partial \nu} = g_i \quad \text{on } \partial B_\delta(a_i) \times (0, \infty) \quad \forall i = 1, \dots, i_0$$

as  $j \rightarrow \infty$ . Letting  $j \rightarrow \infty$  in (3.17),  $u$  satisfies (1.5).

It remains to show that  $u$  has initial value  $u_0$ . For any  $\psi \in C_0^\infty(\Omega_\delta)$ , by (3.17),

$$\begin{aligned}
& \left| \int_{\Omega_\delta} u_j(x, t) \psi(x) dx - \int_{\Omega_\delta} u_{0,j}(x) \psi(x) dx \right| \\
&= \left| \int_0^t \int_{\Omega_\delta} u_{j,t}(x, s) \psi(x) dx ds \right| = \left| \int_0^t \int_{\Omega_\delta} u_j^m(x, s) \Delta \psi(x) dx ds \right| \\
&\leq |\Omega|^{1-m} \|\Delta \psi\|_{L^\infty} \int_0^t \left( \int_{\Omega_\delta} u_j(x, s) dx \right)^m ds \\
&\leq |\Omega|^{1-m} \|\Delta \psi\|_{L^\infty} \left( \int_{\Omega_\delta} u_{0,j}(x) dx + \int_0^t \int_{\partial\Omega} f d\sigma ds + \sum_{i=1}^{i_0} \int_0^t \int_{\partial B_\delta(a_i)} g_i d\sigma ds \right)^m t \quad \forall t > 0.
\end{aligned}$$

Letting  $j \rightarrow \infty$ ,

$$\begin{aligned}
& \left| \int_{\Omega_\delta} u(x, t) \psi(x) dx - \int_{\Omega_\delta} u_0(x) \psi(x) dx \right| \\
&\leq |\Omega|^{1-m} \|\Delta \psi\|_{L^\infty} \left( \int_{\Omega_\delta} u_0(x) dx + \int_0^t \int_{\partial\Omega} f d\sigma ds + \sum_{i=1}^{i_0} \int_0^t \int_{\partial B_\delta(a_i)} g_i d\sigma ds \right)^m t \quad \forall t > 0.
\end{aligned} \tag{3.19}$$

Letting  $t \rightarrow 0$  in (3.19),

$$\lim_{t \rightarrow 0} \int_{\Omega_\delta} u(x, t) \psi(x) dx = \int_{\Omega_\delta} u_0(x) \psi(x) dx \quad \forall \psi \in C_0^\infty(\Omega). \tag{3.20}$$

By (3.20), any sequence  $\{t_k\}_{k=1}^\infty$  converging to 0 as  $k \rightarrow \infty$  will have a convergent subsequence  $\{t_{k_l}\}_{l=1}^\infty$  such that  $u(x, t_{k_l})$  converges to  $u_0(x)$  a.e.  $x \in \Omega_\delta$  as  $l \rightarrow \infty$ . By the Lebesgue Dominated Convergence Theorem

$$\lim_{l \rightarrow \infty} \int_{\Omega_\delta} |u(x, t_{k_l}) - u_0(x)| dx = 0$$

Since the sequence  $\{t_k\}_{k=1}^\infty$  is arbitrary,  $u$  satisfies (1.14).

**Case 2:**  $0 \leq u_0 \in L^\infty(\overline{\Omega_\delta})$ .

We choose a sequence of functions  $\{u_{0,j}\}_{j=1}^\infty \subset C^\infty(\overline{\Omega_\delta})$  such that  $b_j := \min_{\Omega_\delta} u_{0,j} > 0$  on  $\overline{\Omega_\delta}$  for all  $j \in \mathbb{Z}^+$  and

$$\begin{cases} \|u_{0,j} - u_0\|_{L^1(\Omega_\delta)} \rightarrow 0 & \text{as } j \rightarrow \infty \\ u_{0,j}(x) \rightarrow u_0(x) & \text{a.e. } x \in \Omega \text{ as } j \rightarrow \infty \\ \|u_{0,j}\|_{L^\infty(\Omega_\delta)} \leq \|u_0\|_{L^\infty(\Omega_\delta)} + \frac{1}{j} & \forall j \in \mathbb{Z}^+. \end{cases} \tag{3.21}$$

For each  $i = 1, \dots, i_0$ , we choose a sequence of positive functions  $\{g_{i,j}\}_{j=1}^\infty \subset C^\infty(\partial B_\delta(a_i) \times [0, \infty))$  satisfying

$$\begin{cases} g_{i,j} \rightarrow g_i & \text{in } L_{loc}^1(\partial B_\delta(a_i) \times [0, \infty)) & \text{as } j \rightarrow \infty & \forall i = 1, \dots, i_0 \\ g_{i,j}(x, t) \rightarrow g_i(x, t) & \text{a.e. } (x, t) \in \partial B_\delta(a_i) \times [0, \infty) & \text{as } j \rightarrow \infty & \forall i = 1, \dots, i_0 \\ g_{i,j}(x, t) \leq \|g_i\|_{L^\infty(\partial B_\delta(a_i) \times [0, T])} + 1 & \text{on } \partial B_\delta(a_i) \times [0, T) & \forall j \in \mathbb{Z}^+, i = 1, \dots, i_0, T > 0 \\ g_{i,j} = \alpha_{i,j} & \text{on } \partial B_\delta(a_i) \times [0, 1/j) & \forall j \in \mathbb{Z}^+, i = 1, \dots, i_0 \end{cases} \tag{3.22}$$

for some positive constants  $\alpha_{i,j}$  and choose a sequence of nonnegative functions  $\{f_j\}_{j=1}^\infty \subset C^\infty(\partial\Omega \times [0, \infty))$  satisfying

$$\begin{cases} f_j \rightarrow f & \text{in } L^1_{loc}(\partial\Omega \times [0, \infty)) \text{ as } j \rightarrow \infty \\ f_j(x, t) \rightarrow f(x, t) & \text{a.e. } (x, t) \in \partial\Omega \times [0, \infty) \text{ as } j \rightarrow \infty \\ f_j(x, t) \leq \|f\|_{L^\infty(\partial\Omega \times [0, T])} + 1 & \text{on } \partial\Omega_\delta \times [0, T] \quad \forall j \in \mathbb{Z}^+, T > 0 \\ f_j = 0 & \text{on } \partial\Omega \times [0, 1/j] \quad \forall j \in \mathbb{Z}^+. \end{cases} \quad (3.23)$$

If  $f \equiv 0$  on  $\partial\Omega \times (0, \infty)$  and  $g_i$  are nonnegative monotone decreasing function of  $t \in (0, \infty)$  for  $i = 1, \dots, i_0$ , then we choose  $f_j \equiv 0$  on  $\partial\Omega \times (0, \infty)$  for all  $j \in \mathbb{Z}^+$  and the functions  $g_{i,j}$  such that they are positive monotone decreasing functions of  $t \in (0, \infty)$  for  $i = 1, \dots, i_0$  and  $j \in \mathbb{Z}^+$ . Then by case 1 for any  $j \in \mathbb{Z}^+$  there exists a solution  $u_j \in C^{2,1}(\overline{\Omega}_\delta \times (0, \infty))$  of (1.4) in  $\Omega_\delta \times (0, \infty)$  with initial value  $u_{0,j}$  that satisfies

$$\int_{\Omega_\delta} u_j(x, t) dx = \int_{\Omega_\delta} u_{0,j} dx + \iint_{\partial\Omega \times (0, t)} f_j d\sigma ds + \sum_{i=1}^{i_0} \iint_{\partial B_\delta(a_i) \times (0, t)} g_{i,j} d\sigma ds \quad \forall t > 0, j \in \mathbb{Z}^+. \quad (3.24)$$

Let  $t_2 > t_1 > 0$ . Then by (3.21) and Proposition 2.9, there exists a constant  $C > 0$  such that (3.18) holds. Since the constant function  $b_j$  is a subsolution of (1.4) in  $\Omega_\delta \times (0, \infty)$  with  $u_0 = b_j$ . By Lemma 2.1,

$$u_j(x, t) \geq b_j > 0 \quad \forall x \in \Omega_\delta, t \geq 0. \quad (3.25)$$

Then by (3.18) and (3.25) the equation (1.1) is uniformly parabolic for the sequence  $\{u_j\}_{j=1}^\infty$  on every compact subset of  $\partial\Omega_\delta \times (0, \infty)$ . By the Schauder estimates [LSU] (Theorem 3.1 and Theorem 5.4 in chapter 5 of [LSU]), for any compact subset  $K$  of  $\Omega_\delta \times (0, \infty)$ ,

$$\sup_K (|\nabla u_j| + |\partial_{x_k x_l}^2 u_j| + |u_{j,t}|) \leq C_3 \quad \forall k, l = 1, \dots, n, j \in \mathbb{Z}^+ \quad (3.26)$$

and

$$\sup_{(y,s),(y',s') \in K} \frac{|\partial_{x_k x_l}^2 u_j(y, s) - \partial_{x_k x_l}^2 u_j(y', s')|}{|y - y'|^\alpha} + \sup_{(y,s),(y',s') \in K} \frac{|u_{j,t}(y, s) - u_{j,t}(y', s')|}{|s - s'|^{\frac{\alpha}{2}}} \leq C_4 \quad \forall k, l = 1, \dots, n, j \in \mathbb{Z}^+ \quad (3.27)$$

for some constants  $C_3 > 0, C_4 > 0, 0 < \alpha < 1$ .

Hence by the Schauder estimates [LSU] the sequence  $\{u_j\}_{j=1}^\infty$  is equi-Hölder continuous in  $C^{2,1}(K)$  for any compact subset  $K$  of  $\Omega_\delta \times (0, T)$ . Hence by the Ascoli Theorem and a diagonalization argument the sequence  $\{u_j\}_{j=1}^\infty$  has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly in  $C^{2,1}(K)$  for any compact subset  $K \subset \Omega_\delta \times (0, \infty)$  as  $j \rightarrow \infty$  to some function  $u \in C^{2,1}(\Omega_\delta \times (0, \infty))$  that satisfies

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\Omega_\delta} (u \eta_t + u^m \Delta \eta) dx dt + \int_{t_1}^{t_2} \int_{\partial\Omega} f \eta d\sigma dt + \sum_{i=1}^{i_0} \int_{t_1}^{t_2} \int_{\partial B_\delta(a_i)} g_i \eta d\sigma dt \\ & = \int_{\Omega_\delta} u(x, t_2) \eta(x) dx - \int_{\Omega_\delta} u(x, t_1) \eta(x) dx \end{aligned} \quad (3.28)$$

for any  $t_2 > t_1 > 0$ , and  $\eta \in C^2(\overline{\Omega}_\delta \times (0, \infty))$  satisfying  $\partial \eta / \partial \nu = 0$  on  $\partial\Omega_\delta \times (0, \infty)$ . Letting  $j \rightarrow \infty$  in (3.24), by (3.21), (3.22), (3.23) and the Lebesgue Dominated Convergence Theorem, we get that  $u$  satisfies (1.5). By Lemma 3.3,

$$u(x, t) > 0 \quad \forall (x, t) \in \Omega_\delta \times (0, \infty). \quad (3.29)$$

By an argument same as case 1,  $u$  satisfies (1.14). Hence  $u$  is a solution of (1.4).

**Case 3:**  $0 \leq u_0 \in L^p(\Omega_\delta)$ .

For any  $j \in \mathbb{Z}^+$ , let  $u_{0,j}(x) = \min(u_0(x), j)$ . By case 2 there exists a solution  $u_j$  of (1.4) in  $\Omega_\delta \times (0, T_j)$  with initial value  $u_{0,j}$  that satisfies (1.5). Since  $\|u_{0,j}\|_{L^p(\Omega_\delta)} \leq \|u_0\|_{L^p(\Omega_\delta)}$ , by Proposition 2.9 for any  $t_2 > t_1 > 0$  there exists a constant  $C$  such that (3.18) holds. Since  $u_{0,j}$  increases and converges to  $u_0$  as  $j \rightarrow \infty$ , by Lemma 2.1,

$$u_j(x, t) \leq u_{j+1}(x, t) \quad \forall x \in \Omega_\delta, 0 < t < T_j, j \in \mathbb{Z}^+. \quad (3.30)$$

Since  $u_j > 0$  in  $\Omega_\delta \times (0, \infty)$ , by (3.18) and (3.30) the equation (1.1) for the sequence  $\{u_j\}_{j=1}^\infty$  is uniformly parabolic on  $\overline{\Omega_\delta} \times [t_1, t_2]$  for any  $t_2 > t_1 > 0$ . By the Schauder estimates [LSU] (Theorem 3.1 and Theorem 5.4 in chapter 5 of [LSU]), for any compact subset  $K$  of  $\Omega_\delta \times (0, \infty)$ , (3.26) and (3.27) hold for some constants  $C_3 > 0$ ,  $C_4 > 0$ ,  $0 < \alpha < 1$ .

Hence the sequence  $\{u_j\}_{j=1}^\infty$  is uniformly equi-Hölder continuous in  $C^{2,1}(K)$  for any compact set  $K \subset \Omega_\delta \times (0, \infty)$ . Hence by (3.30), the Ascoli theorem, and a diagonalization argument and the sequence  $\{u_j\}_{j=1}^\infty$  increases and converges in  $C^{2,1}(K)$  for any compact set  $K \subset \Omega_\delta \times (0, \infty)$  to a solution  $u \in C^{2,1}(\Omega_\delta \times (0, \infty))$  of (1.1) in  $\Omega_\delta \times (0, T)$ . Putting  $u = u_j$  and  $u_0 = u_{0,j}$  in (1.5), (3.18), (3.28), and letting  $j \rightarrow \infty$ , by (3.18) we get that  $u$  satisfies (1.5), (3.28) and  $u \in L^\infty(\partial\Omega_\delta \times (0, \infty))$ . It remains to show that  $u$  has initial value  $u_0$ .

Let  $t_k \rightarrow 0$  as  $k \rightarrow \infty$ . By the same argument as in the Case 1, the sequence  $\{u(x, t_k)\}_{k=1}^\infty$  has a subsequence which we may assume without loss of generality to the sequence itself such that  $u(x, t_k) \rightarrow u_0$  weakly in  $L^1(\Omega_\delta)$  and a.e.  $x \in \Omega_\delta$  as  $k \rightarrow \infty$ . By the proof of Proposition 2.9, (2.29) holds. Hence  $u(x, t_k)$  converges weakly in  $L^p(\Omega_\delta)$  to some function  $v_0$  as  $k \rightarrow \infty$ . Then there exists a subsequence of  $\{t_k\}_{k=1}^\infty$  which we may assume without loss of generality to be the sequence  $\{t_k\}_{k=1}^\infty$  such that  $u(x, t_k)$  converges to  $v_0(x)$  a.e.  $x \in \Omega_\delta$  as  $k \rightarrow \infty$ . Hence  $v(x) = u_0(x)$  a.e.  $x \in \Omega_\delta$ . Thus

$$\int_{\Omega_\delta} u_0^p dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega_\delta} u^p(x, t_k) dx. \quad (3.31)$$

Letting first  $t \rightarrow 0$  and then  $a \rightarrow 0$  in (2.29),

$$\limsup_{t \rightarrow 0} \int_{\Omega_\delta} u^p(x, t) dx \leq \int_{\Omega_\delta} u_0^p dx. \quad (3.32)$$

By (3.31) and (3.32),

$$\lim_{l \rightarrow 0} \int_{\Omega_\delta} u^p(x, t_k) dx = \int_{\Omega_\delta} u_0^p dx. \quad (3.33)$$

Now consider the function

$$w_k(x) = 2^p(u^p(x, t_k) + u_0^p(x)) - |u(x, t_k) - u_0(x)|^p.$$

Note that  $w_k(x) \geq 0$  on  $\Omega_\delta$  and  $w_k(x) \rightarrow 2^{p+1}u_0^p(x)$  a.e.  $x \in \Omega_\delta$  as  $k \rightarrow \infty$ . Hence by the Fatou Lemma and (3.33),

$$\begin{aligned} 2^{p+1} \int_{\Omega_\delta} u_0^p dx &\leq \liminf_{k \rightarrow \infty} \int_{\Omega_\delta} 2^p(u^p(x, t_k) + u_0^p(x)) - |u(x, t_k) - u_0(x)|^p dx \\ &= 2^{p+1} \int_{\Omega_\delta} u_0^p dx - \limsup_{k \rightarrow \infty} \int_{\Omega_\delta} |u(x, t_k) - u_0(x)|^p dx \\ \Rightarrow \quad \lim_{l \rightarrow \infty} \int_{\Omega_\delta} |u(x, t_k) - u_0(x)|^p dx &= 0. \end{aligned}$$

Since the sequence  $\{t_k\}_{k=1}^\infty$  is arbitrary,  $u$  satisfies (1.14). Hence,  $u$  is a solution of (1.4) in  $\Omega_\delta \times (0, \infty)$ .

If  $f \equiv 0$  on  $\partial\Omega \times (0, \infty)$  and  $g_i, i = 1, \dots, i_0$ , are positive monotone decreasing functions of  $t > 0$ , then by the choice of the approximating functions for  $f$  and  $g_i$  in the construction of solutions of cases 1,2,3 above and Lemma 3.3  $u$  satisfies (1.6) in  $\Omega_\delta \times (0, \infty)$  and the theorem follows.  $\square$

By the same argument as the proof of Theorem 1.1 and Lemma 2.1 we have the following two results.

**Theorem 3.5.** Let  $n \geq 3$ ,  $0 < m \leq \frac{n-2}{n}$ ,  $0 \leq u_0 \in L^p(\Omega)$  for some constant  $p > \frac{n(1-m)}{2}$ , and  $0 \leq f \in L_{loc}^\infty(\partial\Omega \times [0, \infty))$ . Suppose either  $u_0 \not\equiv 0$  on  $\Omega_\delta$  or

$$\int_0^t \int_{\partial\Omega} f \, d\sigma ds > 0 \quad \forall t > 0$$

holds. Then there exists a unique solution  $u$  for the Neumann problem,

$$\begin{cases} u_t = \Delta u^m & \text{in } \Omega \times (0, \infty) \\ \frac{\partial u^m}{\partial \nu} = f & \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \Omega \end{cases}$$

which satisfies

$$\int_\Omega u(x, t) \, dx = \int_\Omega u_0 \, dx + \int_0^t \int_{\partial\Omega} f \, d\sigma ds \quad \forall t > 0.$$

Moreover if  $f \equiv 0$  on  $\partial\Omega \times (0, \infty)$ , then  $u$  satisfies (1.6) in  $\Omega_\delta \times (0, T)$ .

**Corollary 3.6.** Let  $n \geq 3$ ,  $0 < \delta < R$ ,  $0 < m \leq \frac{n-2}{n}$ ,  $p > \frac{n(1-m)}{2}$  and let  $f, g \in L_{loc}^\infty([0, \infty))$  be two nonnegative functions. Suppose  $0 \leq u_0 \in L^p(B_R \setminus B_\delta)$  is a radially symmetric function such that either  $u_0 \not\equiv 0$  on  $\Omega_\delta$  or

$$\int_0^t \int_{\partial B_R} f \, d\sigma ds + \int_0^t \int_{\partial B_\delta} g \, d\sigma ds > 0 \quad \forall t > 0$$

holds. Then there exists a unique solution of

$$\begin{cases} u_t = \Delta u^m & \text{in } (B_R \setminus B_\delta) \times (0, \infty) \\ \frac{\partial u^m}{\partial \nu} = f & \text{on } \partial B_R \times (0, \infty) \\ \frac{\partial u^m}{\partial \nu} = g & \text{on } \partial B_\delta \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } B_R \end{cases}$$

in  $(B_R \setminus B_\delta) \times (0, \infty)$  which is radially symmetric and satisfies (1.5) with  $\Omega = B_R$ ,  $i_0 = 1$  and  $a_1 = 0$ .

#### 4. EXISTENCE OF SINGULAR SOLUTIONS

In this section we will prove the existence of singular solutions of (1.1) in  $\hat{\Omega} \times (0, T)$ .

*Proof of Theorem 1.2.* Let  $\alpha = 2m(q + 4\delta_1^{-2})$  and let

$$0 < \varepsilon_j < \min \left( \frac{\delta_1}{2}, \frac{(1-m)^2 q^2}{(4 + (1-m)q)^2}, \frac{(1-m)q\delta_0}{4 + (1-m)q} \right) \quad \forall j \in \mathbb{Z}^+.$$

be a sequence decreasing to zero as  $j \rightarrow \infty$ . By Theorem 1.1 for any  $j \in \mathbb{Z}^+$  there exists a solution  $u_j$  of

$$\begin{cases} u_t = \Delta u^m & \text{in } \Omega_\delta \times (0, \infty) \\ \frac{\partial u^m}{\partial \nu} = f & \text{on } \partial\Omega \times (0, \infty) \\ \frac{\partial u^m}{\partial \nu} = \frac{\alpha}{\varepsilon_j^{q_m+1}} & \text{on } \partial B_{\varepsilon_j}(a_i) \times (0, \infty) \quad \forall i = 1, \dots, i_0 \\ u(x, 0) = u_0(x) & \text{in } B_R \end{cases} \quad (4.1)$$

By Lemma 2.3 there exists a constant  $0 < \delta_2 < \delta_1$  such that for any  $T > 0$  there exists a constant  $A_1 > 0$  such that

$$u_j(x, t) \leq \phi_{A_1}(x - a_i, t) \quad \forall \varepsilon_j \leq |x - a_i| < \delta_1, \quad 0 \leq t < T, i = 1, \dots, i_0, \varepsilon_j < \delta_2 \quad (4.2)$$

where  $\phi_{A_1}$  is given by (2.4). By Lemma 2.4,

$$u_j(x, t) \geq \frac{C_1}{|x - a_i|^q e^{\frac{1}{\delta_1^2 - |x - a_i|^2}}} \quad \forall \varepsilon_j \leq |x - a_i| < \delta_1, t > 0, \varepsilon_j \leq \frac{\delta_1}{2}, i = 1, \dots, i_0 \quad (4.3)$$

holds for any  $0 < \delta \leq \delta_1/2$ . By Proposition 2.11 for any  $t_2 > t_1 > 0$  and  $\delta' < \delta_1$  there exists a constant  $M_{\delta', t_1, t_2} > 0$  such that

$$u_j(x, t) \leq M_{\delta', t_1, t_2} \quad \forall x \in \Omega_{\delta'}, t_1 \leq t \leq t_2, \varepsilon_j < \frac{\delta'}{2}. \quad (4.4)$$

By (4.4) and Theorem 1.1 of [S] the sequence  $\{u_j\}_{\varepsilon_j < \frac{\delta'}{2}}$  is equi-Hölder continuous on  $\Omega_{\delta'} \times [t_1, t_2]$  for any  $0 < \delta' < \delta_1$  and  $t_2 > t_1 > 0$ . By the Ascoli Theorem and a diagonalization argument the sequence  $\{u_j\}_{j=1}^\infty$  has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly on every compact subset of  $\hat{\Omega} \times (0, \infty)$  to some continuous function  $u$  that satisfies (1.1) in  $\mathcal{D}(\hat{\Omega} \times (0, \infty))$  as  $j \rightarrow \infty$ . Letting  $j \rightarrow \infty$  in (4.3), (4.2), and (4.4), we get (1.9),

$$u(x, t) \leq \phi_{A_1}(x - a_i, t) \quad \forall 0 < |x - a_i| < \delta_1, \quad 0 \leq t < T, i = 1, \dots, i_0 \quad (4.5)$$

and

$$u(x, t) \leq M_{\delta', t_1, t_2} \quad \forall x \in \Omega_{\delta'}, t_1 \leq t \leq t_2. \quad (4.6)$$

By (4.5), (1.10) follows. By (1.9) and Lemma 3.3,

$$u > 0 \quad \text{in } \hat{\Omega} \times (0, \infty).$$

Hence for any  $t_2 > t_1 > 0$  and  $0 < \delta' < \delta_0$ , there exists a constant  $M' > 0$  such that

$$u \geq M' \quad \text{in } \Omega_{\delta'} \times (t_1, t_2). \quad (4.7)$$

By (4.6) and (4.7) the equation (1.1) for  $u$  is uniformly parabolic on  $\Omega_{\delta'} \times (t_1, t_2)$  for any  $t_2 > t_1 > 0$  and  $0 < \delta' < \delta_0$ . Hence by the Schauder estimates [LSU] (Theorem 3.1 and Theorem 5.4 in chapter 5 of [LSU]),  $u \in C^{2+\beta, 1+(\beta/2)}(\hat{\Omega} \times (0, \infty))$  for some constant  $0 < \beta < 1$  is a classical solution of (1.1) in  $\hat{\Omega} \times (0, \infty)$ . Putting  $u = u_j$  in (1.16) and letting  $j \rightarrow \infty$ , we get that  $u$  satisfies (1.16) for any  $0 < t_1 < t_2 < T$  and  $\eta \in C_0^2((\bar{\Omega} \setminus \{a_1, \dots, a_{i_0}\}) \times (0, T))$  satisfying  $\partial\eta/\partial\nu = 0$  on  $\partial\Omega \times (0, T)$ .

Let  $\eta \in C_0^\infty(\hat{\Omega})$  and  $K = \text{supp } \eta$ . Then  $\varepsilon = \text{dist}(K, \partial\Omega \cup \{0\}) > 0$ . Let  $\psi \in C_0^\infty(\hat{\Omega})$ ,  $0 \leq \psi \leq 1$ , be such that  $\psi(x) = 1$  for all  $x \in K$  and  $\psi(x) = 0$  for all  $\text{dist}(x, K) \geq \varepsilon/2$ . Let  $K_1 = \{x \in \Omega : \text{dist}(x, K) \leq \varepsilon/2\}$ .



By the proof of Lemma 3.1 of [HP] there exist constants  $\alpha > 1$  and  $C > 0$  such that

$$\begin{aligned}
& \left| \frac{\partial}{\partial t} \left( \int_{\Omega_{\varepsilon_j}} u_j \psi^\alpha dx \right) \right| \leq C \left( \int_{\Omega_{\varepsilon_j}} u_j \psi^\alpha dx \right)^m \quad \forall t > 0, \varepsilon_j < \varepsilon/2 \\
& \Rightarrow \left( \int_{\Omega_{\varepsilon_j}} u_j(x, t) \psi^\alpha(x) dx \right)^{1-m} \leq \left( \int_{\Omega_{\varepsilon_j}} u_0 \psi^\alpha dx \right)^{1-m} + (1-m)Ct \quad \forall t > 0, \varepsilon_j < \varepsilon/2 \\
& \Rightarrow \int_K u_j(x, t) dx \leq c_{K_1} \quad \forall 0 < t < 1, \varepsilon_j < \varepsilon/2.
\end{aligned} \tag{4.8}$$

where

$$c_{K_1} = \left( \left( \int_{K_1} u_0 dx \right)^{1-m} + (1-m)C \right)^{\frac{1}{1-m}}$$

Hence by (4.8),

$$\begin{aligned}
\left| \int_{\Omega_{\varepsilon_j}} u_j \eta dx - \int_{\Omega_{\varepsilon_j}} u_0 \eta dx \right| & \leq \int_0^t \int_{\Omega_{\varepsilon_j}} u_j^m |\Delta \eta| dx dt \\
& \leq \|\Delta \eta\|_{L^\infty} |K|^{1-m} t \left( \int_K u_j(x, t) dx \right)^m \\
& \leq \|\Delta \eta\|_{L^\infty} |K|^{1-m} c_{K_1}^m t \quad \forall 0 < t < 1, \varepsilon_j < \varepsilon/2.
\end{aligned} \tag{4.9}$$

Letting  $j \rightarrow \infty$  in (4.9),

$$\left| \int_{\Omega} u \eta dx - \int_{\Omega} u_0 \eta dx \right| \leq \|\Delta \eta\|_{L^\infty} |K|^{1-m} c_{K_1}^m t \quad \forall 0 < t < 1. \tag{4.10}$$

Letting  $t \rightarrow 0$  in (4.10) we get (1.15) and Theorem 1.2 follows.  $\square$

We are now ready for the proof of Theorem 1.3.

*Proof of Theorem 1.3.* Let  $R_0 > 3 \max_{1 \leq i \leq i_0} |a_i|$ . Then for any integer  $j > R_0$  by Theorem 1.2 there exists a solution  $u_j$  of

$$\begin{cases} u_t = \Delta u^m & \text{in } \hat{B}_j \times (0, \infty) \\ \frac{\partial u^m}{\partial \nu} = 0 & \text{on } \partial B_j \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \hat{B}_j \end{cases}$$

which satisfies

$$u_j(x, t) \geq \frac{C_1}{|x - a_i|^q e^{\frac{1}{\delta_1^2 - |x - a_i|^2}}} \quad \forall 0 < |x - a_i| < \delta_1, t > 0, i = 1, \dots, i_0 \tag{4.11}$$

and for any  $T > 0$  there exists a constant  $A_1 > 0$  such that

$$u_j(x, t) \leq \phi_{A_1}(x - a_i, t) \quad \forall 0 < |x - a_i| < \delta_1, 0 \leq t < T, i = 1, \dots, i_0, \tag{4.12}$$

where  $\phi_{A_1}$  is given by (2.4). By (4.11), (4.12) and the same argument as the proof of Theorem 1.2 the sequence  $\{u_j\}_{j > R_0}$  has a subsequence that converges to a  $C^{2,1}$  solution  $u$  of (1.11) that satisfies (1.9) and (1.10) for some constant  $C_T > 0$ . This completes the proof of Theorem 1.3.  $\square$

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